Passage of Time for an Astronaut Rotating around a Rotating (Kerr) Black Hole Using Python- An Application of General Theory of Relativity

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Abstract: The General Theory of Relativity (GTR) is a cornerstone of modern physics, revolutionizing our understanding of gravity, depicting it as a curvature of spacetime. Among its solutions, the Kerr equation stands out, describing the spacetime around a rotating mass. This paper delves into exact solutions like the Schwarzschild solution for non-rotating masses and extends to the Kerr solution for rotating black holes. It specifically explores time dilation of an astronaut in the Kerr metric, which is influenced by parameters such as radial distance, angular momentum, and polar angle. The study employs Python for computational analysis, visualizing time dilation through plots that reflect varying black hole rotations and observer positions, offering a nuanced understanding of relativistic effects in extreme gravitational fields.

Keywords: General Theory of Relativity, Schwarzschild Black Hole, Kerr Black Hole, Time Dilation, Astronaut.

1. Introduction

The General Theory of Relativity is a groundbreaking framework in physics conceived by Albert Einstein and published in 1915[1], fundamentally redefining the concept of gravity as curvature in spacetime[2]. It posited gravity as a manifestation of spacetime curvature, influenced by mass and energy, rather than a traditional force[3]. The Einstein Field Equation, central to this theory, is elegantly expressed as:

$$R_{\mu\nu} - 1/2Rg_{\mu\nu} = kT_{\mu\nu} \quad ...(1)$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, and $g_{\mu\nu}$ is the metric tensor and k is a constant, representing the fabric of spacetime. This equation's solutions have been pivotal in our understanding of cosmological and astrophysical phenomena[4]. The first major solution was the Schwarzschild solution of 1918[5], elucidating the spacetime structure surrounding a spherical non-rotating mass, laying the groundwork for black hole physics. In 1972, the Kerr solution emerged[6], detailing the spacetime around rotating masses, and significantly deepening our understanding of rotating black holes. The inherent complexity in solving Einstein's equation arises from its nonlinear characteristics and the challenge of satisfying relevant physical boundary conditions. The introduction of the Newman-Janis Algorithm[7] later offered a transformative approach, converting specific static solutions into their rotating counterparts, thereby demonstrating the profound and intricate relationship between advanced mathematics and theoretical physics in exploring and explaining the complexities of the universe.

The No-hair theorem[8] is a fundamental concept in black hole physics, posits that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterised by only three externally observable classical parameters: mass, electric charge, and angular momentum. This theorem suggests that all other information about the matter that formed a black hole or is falling into it, "disappears" behind the black hole event horizon and is therefore permanently inaccessible to external observers. This paradigm fundamentally challenges the notion of information preservation in the context of astrophysics and quantum mechanics. Recent advancements in astrophysical research have provided compelling evidence in support of the No-hair theorem. Observations from gravitational wave detectors like LIGO and VIRGO[9], studying the mergers of binary black hole systems, have been pivotal. These observations are consistent with the predictions of the No-hair theorem, as the gravitational waves emitted during these events depend solely on the masses and spins of the merging black holes, aligning with the theorem's assertion that these are the only characteristics discernible to an external observer. These empirical findings not only reinforce the No-hair theorem but also enrich our understanding of the fundamental nature of black holes.

2.Methodology

This section discusses the methodology used to conduct the research.

2.1. Aim of the Study

To calculate the time dilation near a Rotating (Kerr) black hole.

2.2. Research Design

A thought experiment: An astronaut revolving around a Kerr Black hole. The time dilation is calculated by keeping the astronaut at different distances and angular displacements from the center of the Black Hole as well as different rotations of the Black Hole.

2.3. Hypothesis

Null hypothesis: There is no change in time flow near a Kerr black hole.

Alternate hypothesis: There is a change in time flow, that is, time dilation near a kerr black hole.

2.4. Tools Used

Google Collab[10], Python [11]

2.5. Data Collection Procedure

The time dilation factor is calculated for three different rotation spin values of black hole. For each spin value, the position of the astronaut is taken to be from 1.01 to 1.1 of the black hole radius and angular displacement from 0 to 90 degree.

3. Discussion

The Schwarzschild solution is derived from Einstein's field equations by setting the stress-energy ($T_{\mu\nu}$) to zero. When using the simplified equation to solve for a non-rotating black hole, the Schwarzschild metric emerges, describing spacetime around a non-rotating, isolated mass.

$$ds^2 = -(1 - \frac{r_s}{r})dt^2 + (1 - \frac{r_s}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \quad ...(2)$$

Line element for Schwarzschild Black Hole

This solution is essential, predicting the existence of black holes and defining the Schwarzschild radius, thus becoming a cornerstone in the understanding of gravitational phenomena and black hole astrophysics[4]. The passage of time for an astronaut orbiting around a Schwazschild black hole is slower than that on Earth [12]. In this study, we will be studying the passage of time around a rotating black hole.

The Kerr solution of black hole is obtained by solving Einstein's Field Equation(EFE)[1], it is also the generalization of the Schwaszchild solution. As Einstein's Field equations are highly non-linear[13], it is difficult to obtain a direct solution of it. The difficulty of obtaining the Kerr solution is more than that of obtaining the Schwaszchild solution. In this paper, we will be showing a trick method to obtain the Kerr solution using the already-known Schwaszchild solution, the method is called the Newman Janis Algorithm.

The Newman Janis algorithm[14] comprises of four steps, the steps are as follows: The first step is the transformation from spherical to a null coordinate {u, r, θ, ϕ}, the second step is to find a null tetrad for the inverse matrix in the null coordinates. The third step is to do a complex transformation in the r-u plane and the last step is the transform of the coordinate into Boyer-Lindquist form.

The line element of the Schwaszchild solution is:

$$ds^2 = -(1 - \frac{r_s}{r})dt^2 + (1 - \frac{r_s}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \quad ...(3)$$

From Cosimo[14], after applying the four steps the final result is:

$$ds^2 = (1 - \frac{2Mr}{\Sigma})dt^2 + \frac{4aMrsin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 -$$

$$sin^2\theta(r^2 + a^2 + \frac{2a^2Mrsin^2\theta}{\Sigma})d\theta^2$$
$$...(4)$$

$$\Sigma = r^2 + a^2cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2 \quad ...(5)$$

Kerr metric for rotating black holes

4. Results

In the study of the Kerr metric, a solution to Einstein's field equations that describes the geometry of spacetime around a rotating mass, the differential elements *dr*, *dθ*, and *dϕ* represent infinitesimal changes in radial distance, polar angle, and azimuthal angle, respectively. Setting these spatial differentials to zero simplifies the metric to analyse scenarios where an observer is fixed in space relative to the rotating mass. This condition focuses on the temporal aspect of spacetime, allowing the evaluation of time dilation effects near a rotating black hole without the complications of spatial movement.The Kerr metric, becomes:

$$ds^2 = (1 - r_s r/\Sigma)dt^2 \ldots (6)$$

under these conditions, where Σ is defined as $r^2 + a^2cos^2\theta$ and encapsulates the effects of the black hole's angular momentum per unit mass (*a*). This specific framework is crucial for analysing how the interplay between the gravitational field and the rotation of the black hole affects the passage of time for a stationary observer at a constant radius and angular position. The result is a cornerstone for understanding the experiential differences in time passage in the strong-field regime of gravity, highlighting the profound effects predicted by general relativity. One can identify the time dilation factor from equation 6 as :

$$\tau = (1 - r_s r/\Sigma)^{1/2}t \ldots (7)$$

$$\tau = (1 - (r_s r/r^2 + a^2cos^2\theta))^{1/2}t \ldots (8)$$

Where, τ and t is the flow of time in the astronauts and Earth's frame of reference. From equation 8, the time dilation depends on the position of the astronauts from the centre of the black hole as well as the angular position with respect to the spin direction of the black hole.

Within the framework of general relativity, the Schwarzschild solution delineates a scenario where time dilation is solely a function of radial distance from a non-rotating spherical mass. In stark contrast, the Kerr solution, applicable to rotating black holes, introduces a more complex dependency: time dilation becomes a function of both the observer's angular displacement and the intrinsic rotation of the black hole. This intricate relationship arises from the drag of spacetime engendered by the black hole's rotation, an effect absent in the Schwarzschild geometry. In the investigation of time dilation within the context of a rotating gravitational field, the time dilation factor is a critical metric that is influenced by three parameters: radial distance r, the spin parameter a, and the polar angle θ. The radial distance r is normalised by the Schwarzschild radius r_s , which is the radius of the event horizon of a non-rotating black hole. The spin parameter a represents the specific angular momentum of the rotating mass, normalised by its mass and the speed of light, encapsulating the effects of rotation on spacetime. The polar angle θ, measured from the object's rotation axis, introduces an anisotropy to the time dilation factor, reflecting the non-spherical symmetry of the spacetime around a rotating mass.

To analyse the time flow for the astronaut around the black hole we will plot and analyse the time dilation factor. Below are the steps taken to obtain the plots:

1. Considering the spin of the black hole to be a = 0.1, 0.5 and 0.7. For each of these spin values, the radial position and angular displacement are varied.
2. The angular position θ is taken as 0, 45 and 90 degrees.
3. The r/rs is taken to be from 1.01 to 1.1.

The below code is a sample code to generate the plots

Code 1: Plot for a = 0.1

```
Import numpy as np
Import matplotlib.pyplot as plt.

rs = 2 # radius of black hole in natural unit
def tau(r, a, theta):
```

```
    t = np.sqrt(1 - ((r/rs)/((r/rs**2) + (a/rs)**2*np.cos(theta **2))
    return t

r_range = np.arange(2.02, 2.22, 0.02)
r_rs = np.range(1.01, 1.11, 0.01)

tau0 = []
tau1 = []
for i in r_range:
    tau0. append(tau(i, 0.1, 0))
    tau1.append(tau(i, 0.1, 90))

plt.scatter(r_rs, tau0, s=5, c='black', label='0 degree')
plt.scatter(r_rs, tau1, s=5, c ='blue', label= '90 degree')
plt.xticks([1.0, 1.01, 1.02, 1.03, 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.10])
plt.xlabel("r/rs")
plt.ylabel("Time Dilation Factor")
plt.title(for spin parameter a= 0.1")
plt.legend()
plt.show()
```

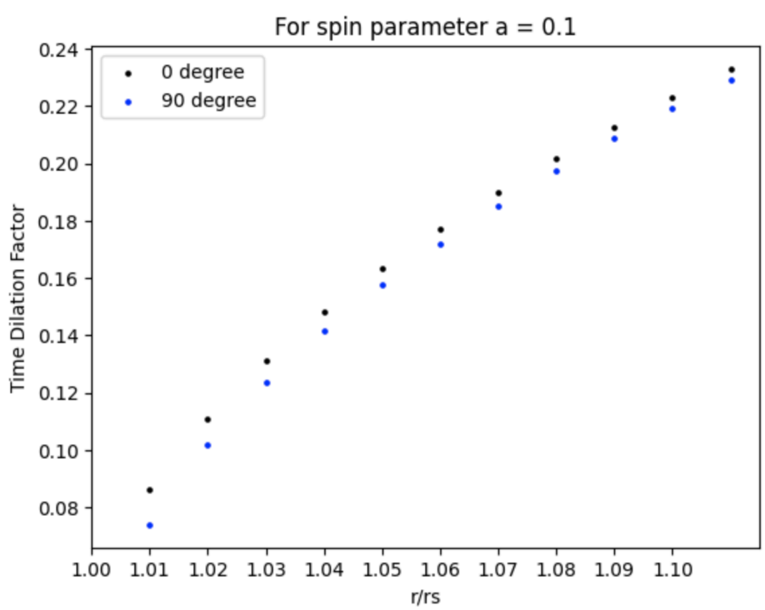

Figure 1: Time dilation factor for spin parameter 0.1

From figure 1 is the plot of time dilation factor for spin parameter 0.1. The radial position is varied from r/rs 1.01 to 1.10 for two different angular positions 0 and 90 degree w.r.t the spinning direction of the black hole.

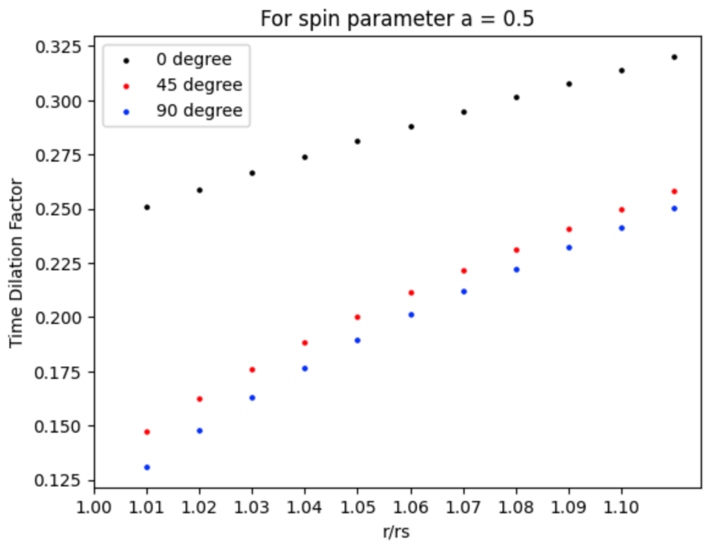

Figure 2: Time dilation factor for spin parameter 0.5

From figure 2 is the plot of time dilation factor for spin parameter 0.5. The radial position is varied from r/rs 1.01 to 1.10 for two different angular positions 0 45 and 90 degree w.r.t the spinning direction of the black hole.

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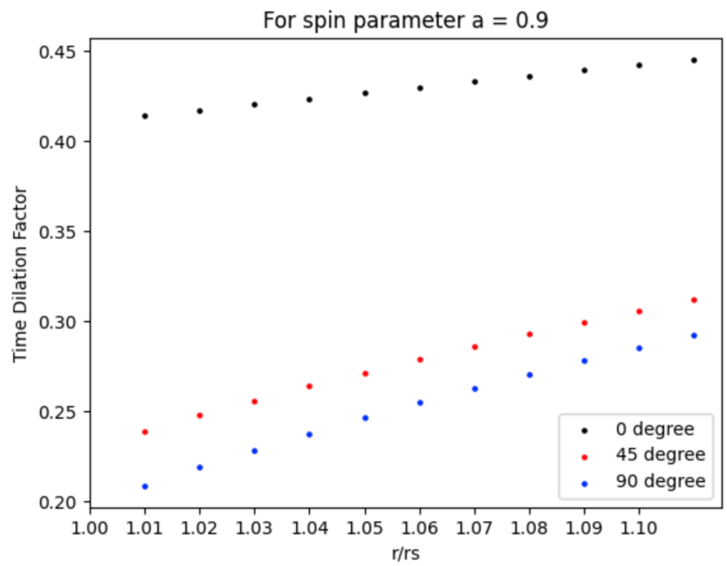

Figure 3: Time dilation factor for spin parameter 0.9

From figure 2 is the plot of time dilation factor for spin parameter 0.5. The radial position is varied from r/rs 1.01 to 1.10 for two different angular positions 0 45 and 90 degree w.r.t the spinning direction of the black hole.

The resulting plots provide a comparative perspective on how time dilation varies not only with proximity to the gravitational source but also with its rotational dynamics. This comparative study yields significant insights into the relativistic effects near rotating celestial bodies and serves to augment our understanding of time dilation in such extreme environments.

5. Conclusion

Upon analyzing the visual data, a clear distinction emerged in the behavior of the time dilation factor relative to the spin parameter *a*. For modest spin parameters, as exemplified by $a = 0.1$, the resultant graph(figure 1) showcased a minimal variation in the time dilation factor with changes in the polar angle θ. This was evidenced by the proximity of the plotted curves for different θ values, suggesting a weaker dependence on the polar angle in spacetimes with slight rotational influence.

Conversely, as the spin parameter increased, the dependency of the time dilation factor on θ became significantly pronounced. This was particularly evident at higher values of *a*, where the graphical representation revealed a marked divergence between the curves corresponding to different θ values. Thus the flow of time for the astronaut orbiting around the Kerr black hole depends on three parameters, radial distance, angular displacement and the spin parameter. From the figure 1, 2, and 3 we can conclude that the passage of time for the astronaut orbiting around the black hole is the slowest when the black hole has a higher spin value, the astronaut is closer to the black hole and the spinning direction.

6. Future Scope

In future scope, this research can be extended to study the time dilation factor for a rotating charged black hole, known as the Kerr-Newman black hole[15]. This exploration would integrate the effects of charge into the already complex interplay of mass and spin in the Kerr metric. Analysing how the presence of an electric charge influences time dilation in the vicinity of such a black hole could provide deeper insights into the nature of spacetime under these extreme conditions.

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