The Proof and Process of Calculating the Geometries for a Rocket Nozzle

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There are many methods to calculate rocket nozzle geometries from reliable sources like NASA, with a necessary proficiency of precalculus/calculus. For most situations, it works, however for those who are only proficient in math to the degree of algebra, there is a lack of solutions. The method that this paper uses takes an expansion ratio equation from [1]. Despite the older publication date, proven by this paper, it still is an effective tool to calculate the said engine hardware geometries. This paper proves the method by first calculating the nozzle throat, and exit areas for a hypothetical cold gas thruster. After these calculations, a physical testing unit is built where after firing, a pressure gauge in the thrust chamber measures the firing(s) performance. After a display of the data in the form of graphs, the algebraically derived predictions will be compared to the data, thereby proving the method. A practical demonstration was chosen to provide sound evidence for the viability of the method. In summary, this paper is not only proof, but also a secondary explanation of the method for designing a rocket nozzle and proof of the viability of a 3D printed nozzle under low pressure.

Introduction

As the topic of space exploration becomes an increasingly prevalent concept, with an entire aerospace based industry blooming in its infancy, education about space travel needs to be more easily accessible at a younger age to those who are interested. Before the 21st century, aerospace engineering was mostly reserved for large government agencies and contractors building for those government agencies often from their designs, and some other companies making small components. Although multiple times, the contractors built the majority of the launch vehicle, it was never truly private until Blue Origin and SpaceX were founded. These companies were the pioneers in the field of commercializing aerospace, and following in their footsteps are about 50 other launch companies who are either developing, proposing, or sometimes even flying orbital launchers today. This doesn't even include the hundreds of companies who construct components for those companies. This could never have been accomplished without the basics learned, such as the rocket nozzle. A rocket nozzle is an important part of a rocket engine that essentially functions to a rocket engine as a car wheel to a car engine, with many different versions of the concept.

Background

One of the most popular rocket nozzle types is the de Laval nozzle, and the nozzle designed in this study. The delaval nozzle works by constricting the flow of the combustion gasses, leading to an increase in speed up to mach 1. After the flow reaches mach 1, the nozzle then expands the gas to speed it up while reducing pressure to the ambient pressure or close to it. This is necessary in order to get the optimal efficiency for the engine as it transfers the energy of the gas/plasma in the combustion chamber into momentum to create the most possible thrust. There are alternative designs that include over and under expanded flows, resulting in different thrust profiles, but the purpose of this study was to examine perfectly expanded flow.

Theoretical Design

This system was designed from the ground up to be a basic and affordable test article for a mathematically derived nozzle. It doesn't have sufficient thrust to lift off with its existing hardware and is just a place to prove or disprove the math. The math behind this is taken from two sources, the first being from Gramlich.net in which all of the articles were authored by Wayne.C.Gramlich [1]. This website was an early website with a simple method for nozzle geometries although a second source was used to make it more accurate. The second source was the NASA thrust equations website [2] along with generic equations used to calculate the geometries of the rocket nozzle. Nasa has a well documented method in a second website however it includes some calculus level math that, as said before, hinders calculation by those of a lower math proficiency. The thrust equation website has equations however, that augment the method described by Gramlich in his website.

Nozzle Design

This section's purpose is to explain the math for the nozzle in simple terms and without calculus level math, as this will enable the comprehension of the method by a large range of ages. To preface this, all units are in metric and will be provided. Also, the subscript "t" means at the throat, and subscript "c" means in the chamber. The most important equation in this method is the throat equation, as it can be rearranged to solve for all of the variables below, and it also is the equation responsible for finding the throat area.:

$$A_t = \frac{\dot{m}}{P_t * \sqrt{\frac{R \cdot T_t}{\Gamma \cdot g_c}}}$$

In this equation, A_t is the throat area in m^2, $\dot{m}$ is the mass flow rate in kg/sec, P_t is the pressure at the throat of the nozzle in pa (pascals), R is the gas constant for the gas in J/kg/K, T_t is the temperature of the throat in K, Γ is Gamma which is the ratio of specific heats in the gas in N/m^3, and finally g_c is the gravitational constant (9.8 meters/sec). This is the equation around which the rest of the method is built around solving.

The second equation is solving for the throat temperature in k (kelvin) from the chamber temperature in kelvin.

$$T_t = T_c\left(\frac{1}{1+\frac{(R-1}{2}}\right)$$

In this equation, the only variable units are kelvin for T_t and T_c, and J/kg/K for R. This next equation can be used to solve for the throat pressure in units of pa (pascals).

$$P_t = P_c\left(1 + \frac{\Gamma-1}{2}\right)^{\frac{-\Gamma}{\Gamma-1}}$$

This equation's only variable unit is pascals for P_t and P_c. The fourth variable is R, which is that gas constant, and this can be found online or can be calculated by equations such as this:

$$R = \frac{(universal\ gas\ constant\ in\ J/kg/K)}{(gas\ molecular\ weight\ in\ g/mol)}$$

This equation is not necessary as detailed before as simply searching it up will give you a satisfactory R. The next equation is used to find the mass flow rate in choked flow. That is the mass flow rate at a part of the pipe that chokes the flow such as a nozzle, with a different equation for finding mass flow rate at any area of a pipe. Here is the representation of the equation.

$$\dot{m} = \frac{A_t * P_t}{\sqrt{T_t}} \cdot \sqrt{\frac{\Gamma}{R}} * \left(\frac{\Gamma+1}{2}\right)^{-\left(\frac{\Gamma+1}{2(\Gamma-1)}\right)}$$

In this equation, A_t is in m^2 as before, P_t is in Pa(pascals), T_t is in K(kelvin), and R in J/kg/K. The mass flow rate is in kg/sec. A method used for this equation to get a good mass flow rate involved getting the choked mass flow rate for the tank drain valve as the mass flow was the most constricted at that point. Also, as recalled from physics, mass flow rate is constant in a system. Finally, gamma is a constant for each gas that is.

All of the necessary variables for the throat are defined at this point. Mass flow rate is defined, as is throat pressure, the gas constant, throat temperature, the gravitational constant, and gamma. Inserting the found values results in a theoretical perfect nozzle throat in m^2 , however there is a second part which is the expansion section. Before proceeding to the exit area equation, the next equation finds the exit mach number.

$$M_e^{\ 2} = \frac{2}{\Gamma-1}\left(\left(\frac{P_c}{P_{atm}}\right)^{\frac{\Gamma-1}{\Gamma}} - 1\right)$$

M_e^2 is the mach number squared which is solved for by variables R inJ/kg/K, P_c in Pa(pascals), and P_{atm} in Pa(pascals) which is the external atmospheric pressure. This following equation only has the throat area in m^2 as a variable along with gamma and the optimal mach number.

$$A_e = A_T\left(\frac{\Gamma+1}{2}\right)^{-\frac{\Gamma+1}{2(\Gamma-1)}} \frac{\left(1+\frac{\Gamma-1}{2}M_e^{\ 2}\right)^{\frac{\Gamma+1}{2(\Gamma-1)}}}{M_e}$$

The variables for this equation are A_t in m^2, and M_e in mach. With this all that is needed are the angles for the converging section's conic cross section as well as the diverging cross section. 45 degrees or 0.785398 radians are commonly cited (Supersonic Wind Tunnel Nozzles[3] paper from NASA) as optimal for the converging section of a nozzle. As for the expansion section, a "bell" is more commonly used compared to a cone due to the "4 to 12 percent" of lost thrust efficiency as quoted in A Performance Comparison of Two Rocket Nozzles by NASA. Here, a cone is used to simplify the math further despite the fact that a bell could be used for higher efficiencies. 15 degrees or 0.261799 radians are optimal for the expansion section for the nozzle for the cone version of the nozzle. With this, a design can be fully illustrated and used.

Now, to move on to the design built for testing the viability of this simplified method. This design had the following parameters with this being a rough overview of the method as well. Keep in mind that some input variables are output variables of previous equations.

Input variables: Throat temperature: T_c=	294.261K
P_c=	689476Pa
R=	$286.765(J/kg/K)$
Γ=	1.401
Inputs for mass flow rate	

$A_{Tank\ Orifice}$= $P_{Tank\ Orifice}$=	$.0001824146900086m^2$ $1307791.09753pa$
Inputs for throat area g_c=	9.8m/s
Mach solution P_{atm}=	101325Pa
Exit Area Solution M_e^2=	1.910 Mach
P_c	689476Pa

Table 1: Input Variables

Equation results:

Outputs: Throat temperature: T_t	287.766
Exit area: A_t	$0.049985505mm^2$
Throat pressure: P_t	364119Pa
Mass flow rate: $\dot{m}$	.9 kg/sec
Throat area: A_t	$0.0319\ mm^2$
Mach speed: M_e^2	1.910 Mach

Table 2: Equation results

Using the process outlined above, the final design for the nozzle was determined. A cross section of the nozzle that was printed may be found below. This shows a cross section of one side

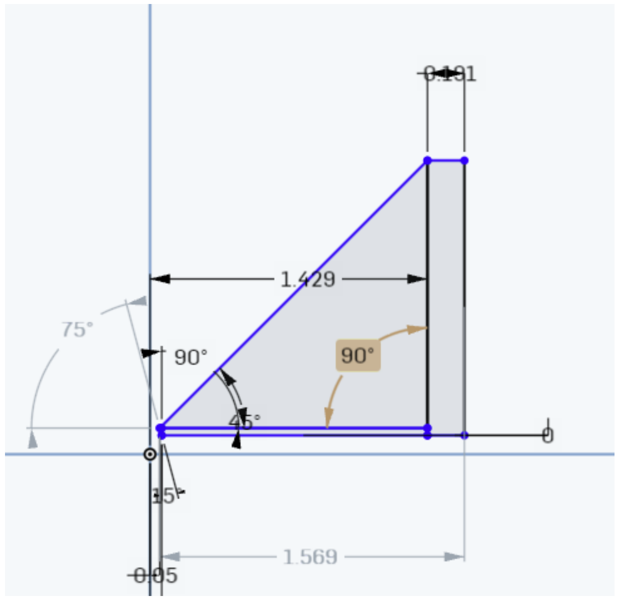

Figure 1: Nozzle Cross Section

Physical Design

As referenced before, the design is built in the name of simplicity and low cost for the experiment. Below is the diagram of the design, which is straightforward in operation.

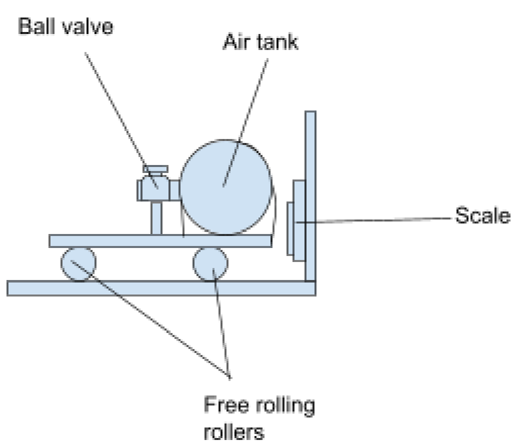

Figure 2: Testing Rig Sketch

The system consists of an air tank connected directly to a valve that is in turn connected to a t-fitting. This includes the 3d printed nozzle in line with the valve, and a pressure gauge perpendicular to the nozzle for chamber pressure measurements. This assembly is fastened to a base that is angled in line with the thrust vector, and the base rolls on free rolling cylinders that are in turn rolling on a second base. Connected to this second base is a vertical mounting wall that can withstand the thrust of the thruster. This wall is positioned behind the system with a scale as the force measuring method mounted on it. The scale is then in contact with the rolling base, where it measures the thrust of the system once every second.

The testing process involves first pressurizing the tank to 100 + psi and then turning on the scale, zeroing it in the process. A recording camera is then turned on with both the chamber pressure gauge and scale in view. Then a rapid manual valve release follows, along with a recording until burn out for a thrust curve mapping.

Physical Results

Below is an image of the testing setup.

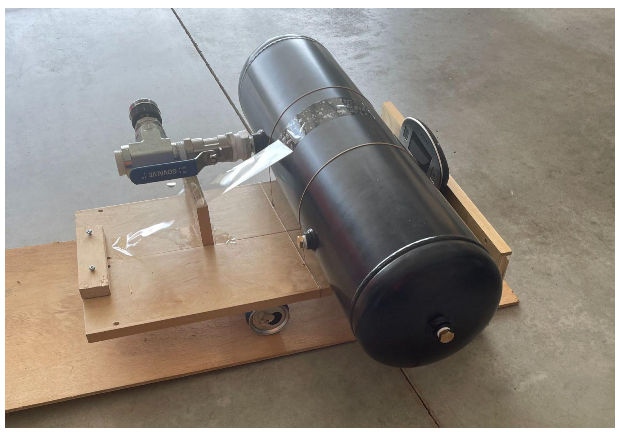

Figure 3: Testing Setup

The temperature on the day of testing was 80 degrees fahrenheit, and the test was conducted on a space without wind for interference. This test was performed a single time due to time constraints for the project, so although the exact percent error will be presented, only the rough error percentage will be considered for the conclusion of whether or not the method is viable or not. The data is presented in a table in the form of (psi, thrust).

Chamber PSI(in intervals of 5)	Thrust in units of Grams
100	135
95	123
90	111.5
85	99
80	88
75	76
70	63

65	54
60	46
55	38
50	33
45	29
40	24.5
35	20
30	14
0	0

Table 3: Testing Results

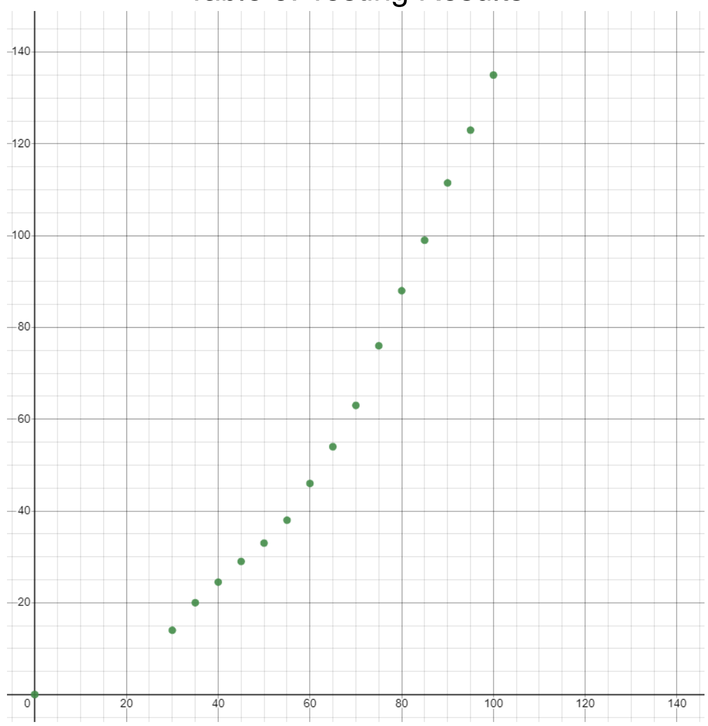

Figure 4: PSI vs Thrust Results

The theoretical thrust for this system was calculated with an equation taken directly from [2]. This equation includes the variables of mass flow rate in kg/sec, V_e in m/s, P_e in Pa(pascals), P_{atm} in Pa(pascals), and A_e in m^2.

$$F = \dot{m}V_e + (P_e - P_0)A_e$$

The theoretical thrust for this engine with its parameters is 129.93 grams of thrust, and the thrust at 100 psi of chamber pressure was 135 grams of thrust. After calculating the percentage error, the system has a 3.93% error from the theoretical thrust.

$$\%error = 100 * \left|\frac{actual\ performance - theoretical\ performance}{theoretical\ performance}\right|$$

This is a relatively low percent error for a system with this scale, and this indicates that the math is accurate to the point of minor external factors being responsible for the percent error. There are a few different factors that led to the difference, such as differing temperature or humidity, as these can't be predicted to a perfect degree. Another possible factor could be an imperfect nozzle size, as although the nozzle was made with as high of an accuracy as possible, at the scale of the test, it is completely possible that the nozzle was slightly off enough to allow for the recorded divergence.

Conclusion

In conclusion, despite the relative complexity of the calculations necessary for advanced and scaled rocket systems, this more accessible method is feasible for smaller systems and those with less strenuous testing requirements. The nozzle after being 3D printed and put on a test stand had thrust measurements in line with the theoretical performance to a percent error of 3.93%. This experiment proves the accuracy of the nozzle design process, the structural integrity of a 3D printed nozzle at low pressures, and the accuracy of a low cost testing platform.

References

[1]Leroy J. Krzycki, HOW to DESIGN, BUILD and TEST SMALL LIQUID-FUEL ROCKET ENGINES,3rd ed, USA: 1999
[2]Tom Benson, Rocket Thrust Summary, 2+ ed, USA: May/13/2021
[3]Stephen W.D. Wolf, Supersonic Wind Tunnel Nozzles, 1st ed, USA: 1990