Modern Portfolio Theory: Variance, Covariance, and the Mathematics of Diversification By Connor Lange

Abstract

This paper explores how Modern Portfolio Theory, introduced by Markowitz, revolutionized investment strategy by adding a mathematical dimension to investment thought. The mathematical structures employed by investors heavily relied on variance and covariance to reduce portfolio risk by maximizing diversification. Central to this is a negative correlation among assets which can largely reduce volatility even if said assets have high risks. In addition, this paper addresses the flaws of MPT, such as performance during market downturns and its reliance on historical data. Alternatives like artificial-intelligence based strategies and Digital Portfolio Theory have an adaptability that MPT lacks. Although MPT's concepts are still very much used and the theory remains foundational, future-oriented investing favors data-responsive and dynamic frameworks.

Introduction

Harry Markowitz, in the late 1940s, was searching for a topic for his doctoral thesis when he had a chance meeting in a waiting room at the University of Chicago's Economics Department. This meeting inspired him to read John Burr Williams' book: "The Theory of Investment". Markowitz was surprised that the book did not address the risk of investments. A few years later, Modern Portfolio Theory, MPT, was introduced in his "Portfolio Selection", a 1952 publication in the *Journal of Finance*. Modern Portfolio Theory is a framework for creating a diversified portfolio that limits the amount of risk assumed by an investor. MPT measures variance and covariance in order to build a diversified portfolio that achieves a position on the efficient frontier. Investors use MPT for creating a portfolio that aligns with the theory's beliefs. For example, investors will likely select assets across various industries in order to advance negative covariance and thus increase diversification. Moreover, investors will use MPT to create portfolios that have the highest expected return for the lowest level of risk. Investors calculate the expected return by dividing each asset's worth by the entire portfolio's value, then multiplying that number by the expected return of the individual asset, and finally adding all the assets together. This enables investors to maximize the expected level of return for a given level of risk. Overall, MPT variables of covariance and variance advance portfolio diversification and lower risk, which remains a core concept in many investment strategies today. The paper will begin with an introduction that explains the background information about MPT and its history. Next, we dive into the basics and the importance of diversification. From there, we will present mathematical proofs and some limits to MPT. Overall, this will outline how covariance and variance are used in formulas to diversify portfolios and reduce risk for investors.

The Foundations and Mathematics of Diversification

Modern Portfolio Theory (MPT), introduced by Harry Markowitz in 1952, revolutionized financial thinking by applying mathematical principles to investment strategies. In Markowitz's seminal paper, *Portfolio Selection*, he proposed that investors should consider how an asset integrates within the entire portfolio, and should not be viewed in isolation. His work argued that investors could minimize risk for a given return and maximize return for a given level of risk with diversification. Markowitz's risk-return optimization set the groundwork for a new era in portfolio management. MPT measures two variables: variance and covariance. Variance, indicated by standard deviation, expresses the risk of a single asset. A higher standard deviation correlates with greater risk as the assets' returns fluctuate more from the mean. On the other hand, covariance measures the relationship between two assets. Assets that move in the same direction have a positive covariance, and assets that move in opposing directions have a negative covariance. Diversification is best advanced by selecting assets with negative correlation, which reduces a portfolio's overall risk. “When assets that are not perfectly positively correlated are combined, the total risk of the portfolio may be less than the weighted average of the individual risks”[7]. This concept of reducing risk with strategic asset combinations and not just picking the safest assets is a core principle of MPT. The efficient frontier, a curve of optimal portfolios with the highest expected return for a given level of risk, portrays this. Portfolios on the frontier are efficient, while portfolios below are suboptimal. MPT constructs the efficient frontier using matrix algebra and optimization techniques to calculate variances, covariances, and expected returns. Zivot models a quadratic function using a variance-covariance matrix to demonstrate portfolio variance[12]. This mathematical structure enables investors to determine optimal asset weights that minimize portfolio volatility. MPT has evolved from a theoretical model into a practical framework within institutional finance. The theory has been widely adopted by endowments, mutual funds, and pension funds, which often utilize software tools based on MPT principles to inform their asset allocation[2]. However, the authors note that the model must be adapted to consider real-world constraints, including liquidity, taxes, and abnormal return distributions—tail events. Although MPT has restraints, it remains foundational in financial education and industry practice. MPT “quantifies tradeoffs between risk and return”[8], teaching students to think systematically about investment decisions. The inclusion of MPT in both economics and computer science courses throughout academia reflects its continued relevance and interdisciplinary influence. MPT's historical development is consistent with changes in market and academic environments.

Markowitz's theory was initially deemed radical but gained traction with the increase in computing power and the shift to data-driven financial modeling. “Subsequent enhancements by Tobin, Sharpe, and others”[6] amplified the MPT’s framework to include concepts like beta (a measure of an assets sensitivity to market movements) and the capital market line (a line representing portfolios that optimally combine the risk-free asset and the market portfolio), which in turn formed the foundation for the Capital Asset Pricing Model (CAPM). MPT was a starting point in financial economics that branched out into decades of innovation. MPT is still used today to determine which asset classes to include in a portfolio and in what proportions. Retail

Investors and institutions apply principles of MPT with tools like ETF portfolios, Monte Carlo simulations, and robo-advisors. Modernized methods combine machine learning, downside risk measures, and behavioral factors. Overall, Modern Portfolio Theory marks a central moment in financial thought - a shift from intuition-based investing to quantitative, risk-aware portfolio construction. It introduces critical principles such as the benefits of diversification, the efficient frontier, covariance, and variance. MPT has evolved from Markowitz's original formulation in 1952 through technological advancement and academic expansion; MPT remains a catalyst for applied finance and theoretical research.

Mathematical Application, Flaws of MPT, and Alternatives

Minimizing variance across individual assets does not lead to an optimally diverse portfolio. Variance or standard deviation expresses the extent to which an asset deviates from its mean and neglects any relation with other assets. Thus, variance cannot be used solely to create optimal portfolios; covariance is also needed. Portfolio construction considers individual asset volatility and how said volatilities interact. Consider that an investor builds a portfolio with low-risk assets. If the assets are highly correlated despite each asset having low risk, the portfolio would have minimal risk reduction. This demonstrates that variance, by itself, is insufficient to ensure diversification, as even portfolios comprising low-variance assets can still exhibit substantial portfolio risk. Thus, covariance is necessary to measure negative correlations and create an optimal portfolio combining asset covariance and variance. This is the mathematical foundation of diversification. However, this foundation rests on the important assumption that variance alone cannot capture risk, and asset returns are distributed normally. Mandelbrot and Taleb argue that substantial losses are more frequent than the MPT predicts, due to real-world asset returns exhibiting asymmetries and fat tails[5] (fat tails refer to extreme events occurring more frequently than expected under a normal distribution). Their arguments illustrate that MPT’s structure remains a strong basis for depicting how diversification reduces volatility, but MPT’s usefulness in turbulent markets is weakened; however, MPT formalizes this with the following formula for a two-asset portfolio's total variance:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2},$$

where w_1 and w_2 represent the portfolio weight of the first and second asset, respectively, σ_1 and σ_2 represent the standard deviations of the first and second assets, respectively, and $\rho_{1,2}$ is the covariance of the two assets. To see diversification in action suppose w_1=.6, w_2=.4, σ_1=.15, σ_2=.10, $\rho_{1,2}$=-.3. Then $w_1^2\sigma_1^2$ =.36x.0225=.0081. $w_2^2\sigma_2^2$ =.16x.01=.0016. $2w_1w_2\sigma_1\sigma_2\rho_{1,2}$= -.00216. Adding these gives a portfolio variance of 0.00745 and a portfolio standard deviation of about 0.0868 or 8.68%, which is a significant drop from either asset alone. Even though the portfolio consisted of individually risky assets, its total volatility was lower because of the negative correlation. This demonstrates that variance alone is insufficient, and asset interactions, as depicted by covariance, are what enable strategic diversification to lower total risk. This formula depicts that risk is dependent on asset variance and covariance. For example, if the third term is negative, expressing a negative covariance among the assets, the overall portfolio risk is lowered. Even if each asset has a high individual variance, a strong negative covariance can considerably reduce portfolio risk. Variance alone cannot lead to such an optimal reduction. For example, suppose an investor holds two risky stocks: one in the oil industry and one in the airline industry. When oil prices rise, airline stocks typically decline due to higher operating costs, indicating a negative covariance. These assets will offset one another's dips and lower the portfolio's volatility through strategic diversification. MPT is often represented using a line graph with the expected return on the y-axis and the standard deviation of each portfolio on the x-axis. The line made represents the efficient frontier and denotes the maximized expected returns for each level of risk. Diversification, achieved through covariance, bends the line outward. This is where we show the results of our experiments. We experimented with a two-asset portfolio to demonstrate these principles. This experiment utilizes simulated returns for two hypothetical assets, A & B, to demonstrate that adding asset B increases the portfolio's diversification and produces a return series with a lower standard deviation than a portfolio consisting of only asset A. We compute the portfolio mean return and standard deviation both empirically (from the return series) and theoretically (using the variance formula to play around with weights, asset standard deviations, and correlation).

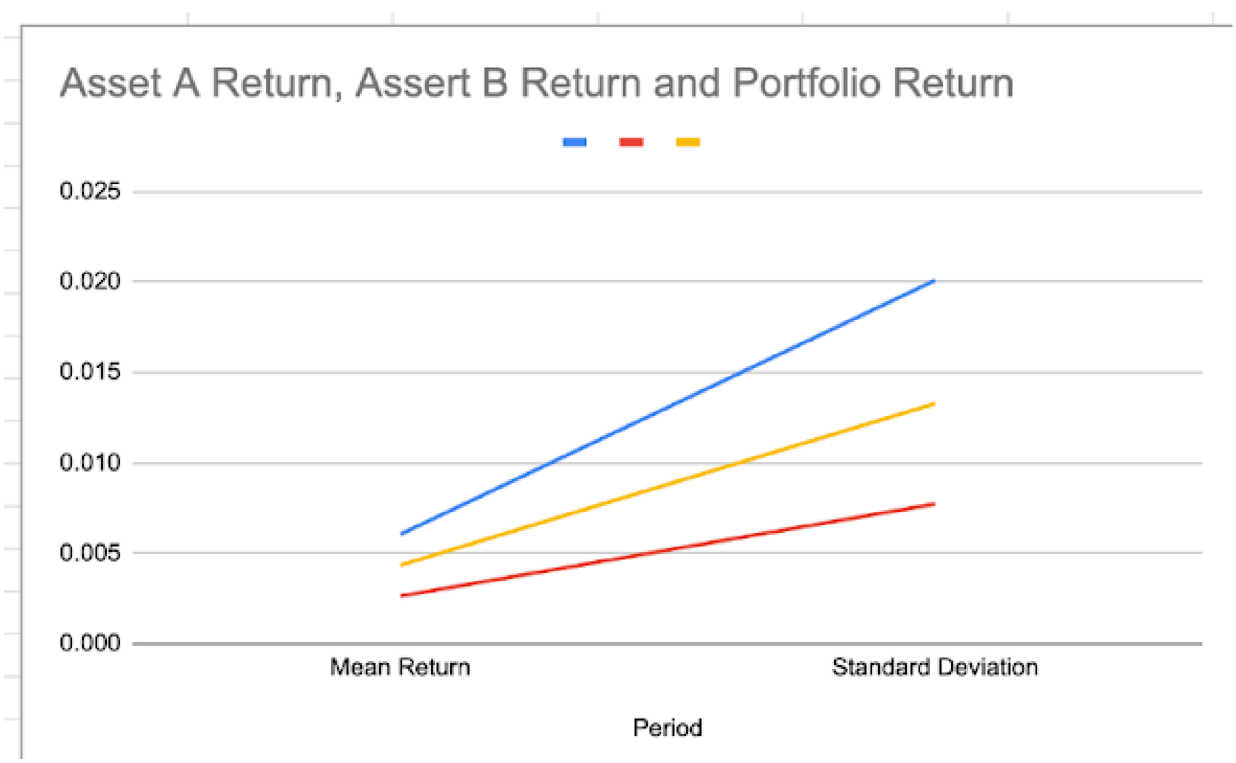

Figure 1. Adding Asset B to the portfolio with negative covariance reduces risk

Figure 1 illustrates how diversification enhances the risk-return profile, thereby supporting MPT. For example, the red line denoting asset A is closest to a vertical line. This means that asset A's mean return in proportion to its standard deviation is less optimal than asset B's proportions, for example, whose red line is closer to being horizontal. However, adding asset B into said portfolio creates the yellow line, which denotes a significant change from the initial blue line. The yellow dual-asset portfolio is closer to a horizontal line than the original, which only includes asset A. In other words, the addition of asset B lowered the standard deviation of the portfolio as a whole.

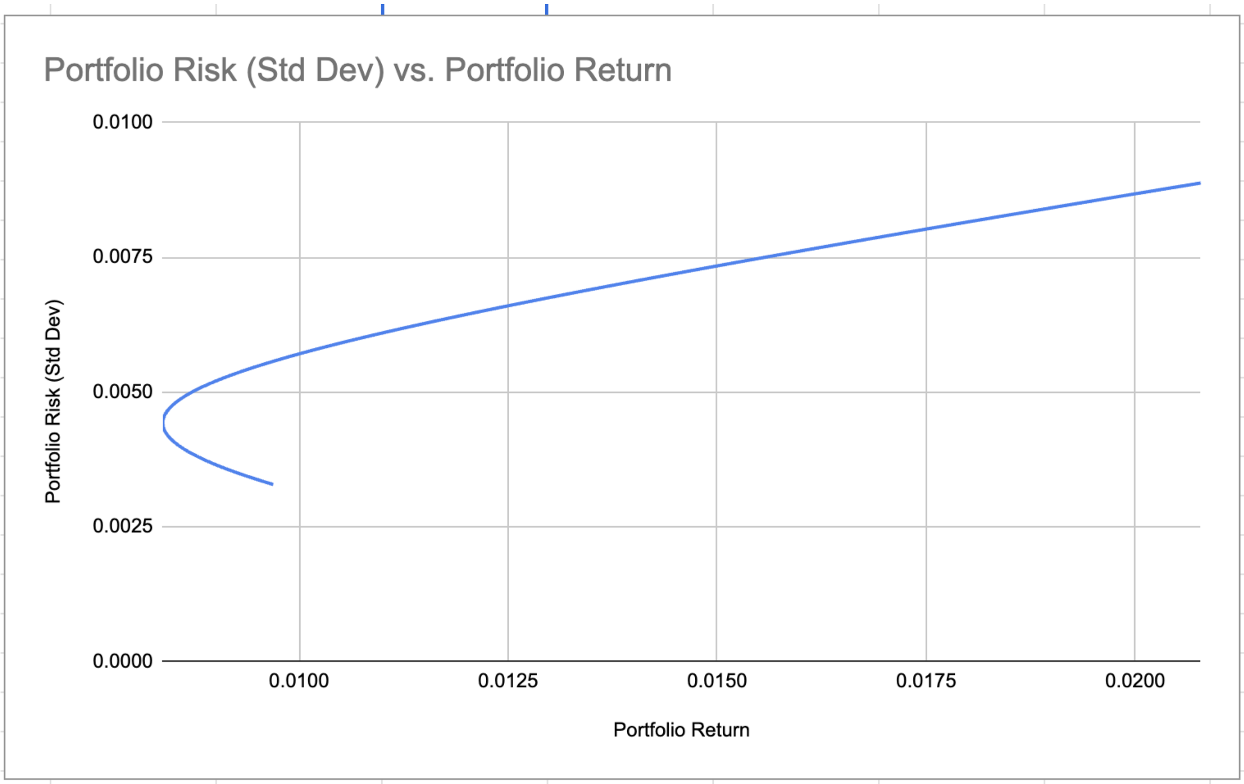

Figure 2. Risk vs return

Figure 2 shows the optimal balance of the portfolio consisting of assets A and B balanced against risk.

MPT makes unrealistic assumptions regarding investor rationality, market behavior, and risk measurement. These flaws are a result of expert analysis and scholarly studies, highlighting a constraint with MPTs' usefulness. A central flaw of MPT is its use of historical data to predict future asset variances, covariances, and returns. This is an issue because markets are constantly evolving, meaning that the relationship between assets is in a state of constant flux. Overall, this weakens diversification as it leads to unstable correlation matrices.

According to Andrew Lo in his Adaptive Market Hypothesis, “The assumption of stationarity in return distributions is both empirically and theoretically fragile”[7]. During the dot-com crash from 2000 to 2002, Lo observed that the average pairwise correlations between sectors, such as industrial and technology, climbed from 0.3 to 0.6, significantly increasing systematic portfolio risk. In a 50-asset portfolio, despite any changes to individual volatilities, this shift was responsible for doubling portfolio variance. Lo denotes a prime example of issues that arise when relying on historical data. Similarly, Didier Sornette observes that climbing correlations precede significant dips as a result of internal feedback loops, essentially identifying endogenous crashes[10] (market crashes caused by internal dynamics rather than external shocks). According to his research, in the month before crashes, correlations rose from approximately 0.25 to 0.6-0.75, which in turn increased portfolio risk by up to 41%. Investors who relied on historical correlations would end up severely underestimating the risk exposure at a critical time. Secondly, MPT often struggles to explain crashes. The MPT principle of diversification assumes that asset correlations are low and stable. However, during market crashes, this is quite the contrary, as correlations behave erratically. As Wysocki explains, “During periods of crisis, correlations between asset returns tend to converge, making diversification less effective just when it is needed most”[11]. This convergence highlights MPT's inability to handle crashes, as every asset class declines while diversification decreases, creating a sensitive situation for MPT portfolios during market panics. MPT variables of covariance and variance can be challenging to calculate in relation to the investor's needs, which can serve as another potential flaw. For example, MPT typically evaluates risk through variance, which considers upside and downside risk fluctuations to be equal; however, many investors care more about downside risk than upside volatility. The University of Washington's portfolio theory notes state that “variance does not distinguish between upside and downside risk, even though investors care much more about losses than gains”.[8] Moreover, covariance estimation can be difficult and unstable, particularly as portfolios grow in size[4]. As Ledoit and Wolf explain, traditional covariance matrices are unstable in high-dimensional settings, prompting the need for shrinkage techniques to improve estimation accuracy [3]. These issues can cause optimization errors and overfitting(when a model is too closely fitted to historical data and performs poorly on new data) problems. Gupta shows how investors can use risk-reward ratios such as Sterling and Treynor to improve portfolio construction, especially in changing markets, suggesting that MPT can be adjusted for dynamic environments [1].

In light of MPT flaws, a more modern theory, such as Digital Portfolio Theory (DPT), as introduced by Jones[2], could be a viable alternative. DPT extends MPT and introduces dynamic memory-based adjustments. DPT assumes that market behavior evolves and needs models to adapt accordingly, whereas MPT focuses on a static optimal weight assumption. DPT treats assets as time-varying functions. According to the Yale University Computer Science 458 lecture materials: “Digital portfolio theory treats asset weights as time-varying functions, responding to regime shifts rather than assuming static optimization”[10]. DPT bolsters responsiveness to sudden market changes, such as the correlation shifts described in Lo[5] and Sornette[9], recalibrating portfolios accordingly. DPT does not assume that asset relationships remain fixed; instead, it monitors them over time. Furthermore, DPT integrates adaptive models, rolling covariance matrices, and real-time data streams in contrast to MPT, which regards periods as statistically identical and relies on historical averages.

Another technique that has the potential to surpass portfolio theories is portfolio construction with machine learning and Advanced AI. AI/ML models help to identify non-linear patterns, avoid overfitting, and adapt to regime shifts. AI systems that learn from high-dimensional market data can respond to market shocks or changing correlations more effectively. Techniques such as random forests can detect non-linear, multi-factor interactions; neural networks learn structural relationships and have high adaptivity; and reinforcement learning uses simulated environments with a trial-and-error approach to optimize allocation strategies. Additionally, the speed at which AI calculates would potentially boost the concept of day by dynamically managing risk and lowering latency.

Conclusion

In Conclusion, Modern portfolio theory has been fundamental to finance for understanding diversification and the risk-return relationship. MPT revolutionized thinking around asset allocation by focusing on optimizing covariance and variance. The application of mathematical models to quantify portfolio risk allowed a more straightforward approach to diversification. Despite all the success MPT has endured and further facilitated, it has apparent flaws that stand out more in the modern market: the quantification of variance and covariance, its reaction to market crashes, and reliance on historical data. Variance cannot solely capture a portfolio's risk; in fact, asset correlation is one of the most important considerations when examining risk, as it can cause the most devastating effects. Los adaptive market hypothesis and Sornette's endogenous study provide empirical evidence demonstrating how markets remain in constant flux, responding to changing conditions and evolving over time. This emphasizes the need for dynamic models that challenge MPT's dynamic approach. Alternatives, such as, Digital Portfolio Theory incorporate optimization with rebalancing based on real data streams. Moreover, AI can manage high-dimensional data and find non-linear patterns. Overall, both approaches offer more realistic and market-aware investment strategies than MPT does. However, continued research and exploration can help refine these practices and further advance investment strategies and analysis. In sum, while MPT has led to groundbreaking discoveries and brought important financial topics to light, today's market requires a more flexible model that assesses the future.

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