How can group theory be used to explain symmetries in classical music?

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1 Background

1.1 Post-tonal Theory

In Western music, there are 12 tones in an octave.

Definition 1.1. *A pitch class is a set of pitches with the same or enharmonic name, and they are represented by an integer from 0 to 11.[17]*

By convention, C is 0, $C\#$ is 1, . . . , B is 11. This pitch system uses modulo 12, so any pitch x is equivalent to $x + 12k$, where k is an integer. We use modulo 12 because there are 12 semitones in an octave, and we consider pitches that are separated by octaves to be equivalent.[17]

Definition 1.2. *A pitch class set is an unordered collection of pitch classes.[17]*

Definition 1.3. *Let $P := \{0, 1, 2, \ldots, 11\}$ be the set of pitch classes.*

The pitch class 0 represents C, 1 represents $C\#$, 2 represents D, and so on.

Definition 1.4. *The transposition of a pitch class p by n is equal to $(p + n) \ mod \ 12$.[17] This defines a map $T_n : P \to P$.*

Example 1.5. *The transposition of 3 by 5 is 3 + 5 = 8. The transposition of 9 by 7 is 9 + 7 = 16 ≡ 4 (mod 12).*

The only transposition that keeps a pitch class the same is the transposition by 0.

Definition 1.6. *The inversion of p around n is equal to $(n - p) \ mod \ 12$.[17] This defines a map $I_n : P \to P$.*

Example 1.7. *The inversion of 3 around 8 is 8 − 3 = 5. The inversion of 8 around 3 is $3 - 8 = -5 \equiv 7 \ (mod \ 12)$.*

The only inversion that keeps a pitch class the same is the transposition by 2 times the pitch class.

Example 1.8. *The inversion of 3 around 2 × 3 = 6 is 6 − 3 = 3. The inversion of 6 around $2 \times 6 = 12 \equiv 0 \ (mod \ 12)$ is $0 - 6 = -6 \equiv 6 \ (mod \ 12)$.*

1.2 Group Theory

1.2.1 Group

Definition 1.9. A *group is a set with a binary operation. Let* G *be a group with an operation* $\cdot$*. Then* G *must follow three group axioms. The first axiom is associativity: for all* $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$*. The second axiom is that there is a unique identity element* $e \in G$ *such that for every* $a \in G$, $e \cdot a = a$ *and* $a \cdot e = a$*. The third axiom is that for every element* $a \in G$ *there exists an element* $b \in G$ *such that* $a \cdot b = e$ *and* $b \cdot a = e$*. [2]*

Groups can be used to describe the symmetries of geometric objects.

Example 1.10. *One example is the dihedral group, which is the rotational and reflectional symmetries of a regular polygon.[6] This group can also be thought of as the permutations of a polygon's vertices, and therefore it is a permutation group.*

Definition 1.11. *The degree of a permutation group is the number of elements of the set being permuted.[9]*

Definition 1.12. *A presentation of a group is a set of generators and a set of relations among the generators that completely describe the group.[10] Every element of the group can be represented as a product of generators. It is expressed as* $\langle S|R\rangle$ *where* S *is a set of generators and* R *is a set of relations among the generators.*

Definition 1.13. *The order of a group* G *is the number of elements it has and is denoted* $|G|$*. The order* m *of an element* a *in that group is the smallest positive integer such that* $a^m = e$ *where* e *is the identity element of* G *and*

$$a^m = \underbrace{aaa...}_{m\ times}$$

means the element a *operated with itself* m *times.[3]*

Definition 1.14. *A subgroup is a subset* H *of group elements of a group* G *that satisfies the group axioms. This is written as* $H \subseteq G$*.[20]*

1.2.2 Homomorphism and Isomorphism

Definition 1.15. *A group homomorphism is a function* $f : G \to H$ *between two groups* G *and* H *such that the group operation is preserved, meaning that* $f(ab) = f(a)f(b)$ *for all* $a, b \in G$*.[13]*

Definition 1.16. *A group isomorphism is a bijective correspondence between the set of all elements of group G and the set of all elements of group H if the group operation is preserved. That is, a relation of the form $g_1g_2 = g_3$ being true means that $h_1h_2 = h_3$ is also true, where $h_1, h_2, h_3 \in H$ are the corresponding elements of $g_1, g_2, g_3 \in G$.[2][16]*

If two groups G and H are isomorphic, then we can write $G \cong H$. Proving two groups are isomorphic is important because we can treat them as essentially the same since they have the exact same group structure, the only difference is what we call the groups and their elements.

Definition 1.17. *Let $f: A \to B$ be a group homomorphism. The set of all elements $x \in A$ that are mapped by f into the identity element of B is called the kernel of f.[2] This is denoted as $ker(f)$.*

1.2.3 Group action

Definition 1.18. *A group action $\phi : G \times X \to X$ [1] is a map such that for all elements $x \in X$, $\phi(e, x) = x$ where e is the identity element of G, and $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$. [12]*

Definition 1.19. *The orbit of an element $x \in X$ under a group action ϕ of group G on set X is $Orb_\phi(x) = \{gx \in X : g \in G\}$, the set of elements of X that is reached by any element of G acting on x.[14]*

Definition 1.20. *The stabilizer of an element $x \in X$ under a group action ϕ of group G on set X is $Stab_\phi(x) = \{g \in G : gx = x\}$, the set of group elements that send x to itself. [19]*

Definition 1.21. *A group action $\phi : G \times X \to X$ acting on a subset $S \subseteq X$ means that $gS = \{gs : s \in S\} \subseteq X$.*

Definition 1.22. *The orbit of a subset $S \subseteq X$ under a group action ϕ of group G on set X is $Orb_\phi(S) = \{gS \subseteq X : g \in G\}$, the set of subsets of X that is reached by any element of G acting on S.*

Definition 1.23. *The stabilizer of a subset $S \subseteq X$ under a group action ϕ of group G on set X is $Stab_\phi(S) = \{g \in G : gS = S\}$, the set of group elements that send S to itself.*

[1]The action inputs a group element of G and a set element of X and outputs a set element of X

Definition 1.24. *Let H be a subgroup of the group G and let $x \in G$. Define xH to be the set $\{xh : h \in H\}$ and Hx to be the set $\{hx : h \in H\}$. A subset of G in the form xH for some $x \in G$ is a left coset of H and a subset in the form Hx is a right coset of H.[5]*

Theorem 1.25. *(Orbit-Stabilizer Theorem) Let G be a finite group acting on a set X with the group action ϕ, and let $x \in X$. Then the number of elements in $Orb_{\phi}(x)$ is equal to the number of distinct left cosets of $Stab_{\phi}(x)$ in G.[1]*

Definition 1.26. *A group action $\phi : G \times X \rightarrow X$ is free if, for all $x \in X, gx = x$ implies $g = e$ where e is the identity element of G.[11]*

Definition 1.27. *A group action $\phi : G \times X \rightarrow X$ is transitive if there is a group element g such that $gx = y$ for all $x, y \in X$.[15]*

2 Transformation and D_{12} Groups

Definition 2.1. *Let $D_{12} := \{O_0, O_1, O_2, \ldots, O_{11}, E_0, E_1, E_2, \ldots, E_{11}\}$ be the dihedral group of degree 12. This group represents the rotational and reflectional symmetries of a regular dodecagon (Figure 1). Here, O_n represents a rotation by n points. Let the corners of the dodecagon have labels 0, 1, 2, . . . , 11. If n is even, then E_n represents the symmetric reflection over a line that passes through the corner $n/2$. If n is odd, then E_n represents the reflection over a line that passes through the midpoint of the side connecting $\frac{(n-1)}{2}$ and $\frac{(n+1)}{2}$.*

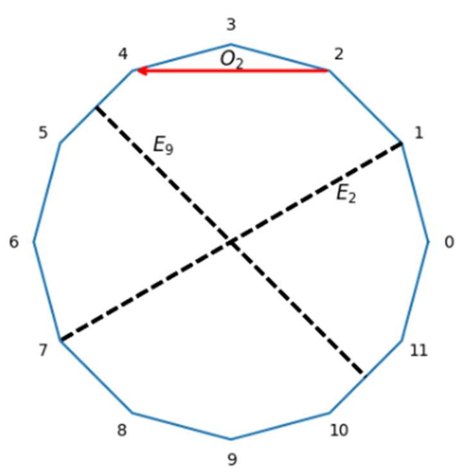

Figure 1: A dodecagon with labeled points and example symmetries of D_{12}.

D_{12} has the presentation $\langle O_1, E_0 | ord(O_1) = 12, ord(E_0) = 2, E_0 O_1 E_0^{-1} = O_1^{-1} \rangle$.

Definition 2.2. *Let $T := \{T_0, T_1, T_2, \ldots, T_{11}, I_0, I_1, I_2, \ldots, I_{11}\}$ with the operation of function composition be the transformation group, where T_n is the transposition by n and I_n is the inversion around n.*

2.1 Transformation and D_{12} Groups Are Isomorphic

Lemma 2.3. *If $f : G \to H$ is a surjective group homomorphism and $|G| = |H|$, then f is an isomorphism.*

Proof. f is surjective, so the image of f, $Im(f) = H$. By the first isomorphism theorem, also known as the fundamental theorem on homomorphisms[4],

$$G/\,ker(f) \;\cong\; Im(f) = |H|$$

Therefore

$$|G/\,ker(f)| = |H|$$
$$\frac{|G|}{|ker(f)|} = |H|$$

Since $|G| = |H|$,

$$\frac{|H|}{|ker(f)|} = |H|$$
$$|ker(f)| = 1$$

The kernel of f is only the identity, so f is injective.[7] Therefore, f is an isomorphism.

Theorem 2.4. *T is isomorphic to D_{12}.*

Proof. T is generated by T_1 and I_0. $\mathrm{ord}(T1) = 12$ and $ord(I_0) = 2$. $I_0T_1I_0$ on x is $0 - (0 - x + 1) = x - 1 \equiv x + 11 \ (mod\ 12)$ and $T_1^{-1} = T_{11}$. We have shown that T has the generators T_1 and I_0 and the relations $\mathrm{ord}(T1) = 12$, $ord(I_0) = 2$, $I_0T_1I_0 = T_1^{-1}$.

The map $f : T \to D_{12}$ maps the generators and relations of T to the generators and relations of D_{12}, so f extends to a unique surjective group homomorphism. Both T and D_{12} have 24 elements, so $|T| = |D_{12}|$.

Thus, $T \cong D_{12}$ by Lemma 2.3

2.2 Transformation as a Group Action

Definition 2.5. *Let $Sym(P)$ be the symmetric group of P consisting of all permutations of the elements of P .[21]*

Definition 2.6. *Let the set B be a subset of the set A. Then the injection $f : B \rightarrow A$ that has the formula $f(b) = b$ for all $b \in B$ is called the inclusion map.[18]*

Definition 2.7. *Let $t : T \rightarrow Sym(P)$ be the group action of transformation.[2] This action takes in a transformation in T and expresses it as a permutation of the pitch classes in P. If $T \subseteq Sym(P)$ is a subgroup, then $t : T \rightarrow Sym(P)$ is an inclusion map with formula $t(g) = g$ for $g \in T$.*

$Orb_t(p) = P$ for all $p \in P$ because for any two pitch classes $p_1, p_2 \in P$, the element T_{p2-p1} of group T maps $p1$ to $p2$.

Only $n = 0$ satisfies $p + n \equiv p \bmod 12$, so T_0 is the only transposition in $Stab_t(p)$. Only $n = 2p \bmod 12$ satisfies $n - p \equiv p \bmod 12$, so $I_{2p \bmod 12}$ is the only inversion in $Stab_t(p)$. $Stab_t(p)$ is the set $\{T_0, I_{2p}\}$ for all $p \in P$.

Under the isomorphism proved in Theorem 2.4, *t* is the action of D_{12} on a dodecagon by left multiplication which is always free and transitive.

2.3 D_{12} Group Acting on the Set of Pitch Classes

Definition 2.8. *The trajectory of a group element g on the set element $x \in X$ is the orbit $Orb_\phi(x)$ under the action $\phi : \langle g \rangle \times X \rightarrow X$ where $\langle g \rangle$ is the cyclic subgroup of g, a group that is generated by only g.*

The trajectory of O_0 on p is $\{p\}$, since rotating by 0 leaves everything the same.

If $n = 1, 5, 7, \text{or } 11$, then the trajectory of O_n on p is equivalent to $P = \{0, 1, 2, \ldots, 11\}$. These four numbers are relatively prime to $0 \equiv 12\ (mod\ 12)$, so the smallest positive integer k for which $kn \equiv 12$ is 12.

If $n = 2$ or $n = 10$, the trajectory of O_n on p is determined by the parity of p. Specifically:

- If p is even, the trajectory is {0, 2, 4, 6, 8, 10}.
- If p is odd, the trajectory is {1, 3, 5, 7, 9, 11}.

This is because O_2 rotates p to each number with the same parity. O_{10} rotates p by 10, which is equivalent to rotating backwards by 2 since 10 ≡ −2.

The trajectory of O_3 and O_9 on p is {0, 3, 6, 9} if $p \equiv 0, \{1, 4, 7, 10\}$ if $p \equiv 1$, and {2, 5, 8, 11} if p ≡ 2.

[2] This is equivalent to $t : T \times P \rightarrow P$

The trajectory of O_4 and O_8 on p is {0, 4, 8} if $p \equiv 0, \{1, 5, 9\}$ if $p \equiv 1, \{2, 6, 10\}$ if $p \equiv 2$, and {3, 7, 11} if $p \equiv 3$.

The trajectory of O_6 on p is $\{p, p + 6 \text{ (mod } 12)\}$.

Let's consider E_0. The point 0 goes to 0, 1 to 11, 2 to 10. 6 goes to 6, 7 goes to 5, 8 goes to 4. We see that for point p and its reflection $q, p + q \equiv 0$. Thus, the trajectory of E_0 on p is $\{p, -p \text{ (mod } 12)\}$.

Now, let's consider E_1. The point 0 goes to 1, 1 goes to 0, 2 to 11, 3 to 10. 6 goes to 7, 7 goes to 6, 8 goes to 7. In this case, $p + q \equiv 1$. Thus, the trajectory of E_1 on p is $\{p, 1 - p \ (mod\ 12)\}$.

In general, the trajectory of E_n on p is $\{p, n - p \ (mod\ 12)\}$.

3 Symmetries in Classical Music

3.1 Diatonic Set Class

Definition 3.1. *The trajectory of a group element g on the subset $S \subseteq X$ is the orbit $Orb_\phi(S)$ under the action $\phi : \langle g \rangle \times X \rightarrow X$.*

Definition 3.2. *A T_n-type is a set consisting of a pitch class set X and all of the pitch class sets that can be reached by applying T_1 repeatedly on X. In other words, the trajectory of T_1 on X.*

In a diatonic scale there are 7 pitch classes.

Definition 3.3. *Let $D \subseteq P$ be the 7 diatonic pitches {0, 2, 4, 5, 7, 9, 11}.*

Definition 3.4. *The diatonic T_n-type is a T_n-type specifically with the set D: $\{1, 3, 5, 6, 8, 10, 12 \equiv 0\}, \{2, 4, 6, 7, 9, 11, 13 \equiv 1\}, \ldots, \{11, 13 \equiv 1, 15 \equiv 3, 16 \equiv 4, 18 \equiv 6, 20 \equiv 8, 22, \equiv 10\}$.*

We can think of the diatonic T_n type as containing all 12 major scales.

The translates gD of D for $g \in T$ are all 12 of the major scales. $gD = D$ when $g = T_0$ or $g = I_4$, so $Stab_t(D) = \{T_0, I_4\}$.

Now, let's find $I_4(D)$. $I_4(\{0, 2, 4, 5, 7, 9, 11\}) = \{0, 2, 4, 5, 7, 9, 11\}$. $I_4(D) = D = T_0(D)$. Is it true that any inversion of D is a transposition of D?

$$I_x(D) = \{x, x - 2, x - 4, x - 5, x - 7, x - 9, x - 11\}$$

$$= \{x + 12, x + 10, x + 8, x + 19, x + 17, x + 15, x + 13\}\ by\ congruence\ mod\ 12$$

$$= \{x + 8, x + 10, x + 12, x + 13, x + 15, x + 17, x + 19\}\ rearranging$$

$$= \{(x+8)+0, (x+8)+2, (x+8)+4, (x+8)+5, (x+8)+7, (x+8)+9, (x+8)+11\}$$

$$= T_{x+8}(D)$$

Definition 3.5. *A T_n-type A is the inverse of the T_n-type B when any pitch class set in A can be mapped to any pitch class set in B by some I_n.*

Therefore, the diatonic T_n-type is the inverse of itself.

Definition 3.6. *A set class is a pair of T_n-types that are inverses of each other.*

$Stab_t(D)$ has order 2, so a set class can be partitioned into two T_n-types by the Orbit-Stabilizer Theorem (Theorem 1.25). Any set X with a group action of T can be written as a union of two of these orbits.

Definition 3.7. *The diatonic set class is the set class with the pair of T_n-types being two copies of the diatonic T_n-type.*

3.2 Diatonic Symmetry Group

Using the dodecagon structure we had for the T group, we pick the points that correspond to a diatonic scale: 0, 2, 4, 5, 7, 9, 11. Then we define a ”forgetting” operation that removes the dodecagon structure, so that there are no known distances between points. When this operation acts on the T group, the structure now only has seven points with an ordering. We can assume that the points lie on a circle and are the same distance apart, and so they form a heptagon. There is no reason to use the original numbers as labels, so we can now just use 0 to 6.[3]

Musically, this is like only looking at the white keys of a piano. The interval between C and E is now the same as the interval between D and F : 2 keys apart.

Definition 3.8. *Let $P_d := \{0, 1, 2, 3, 4, 5, 6\}$ be the set of diatonic pitch classes where 0 is the tonic, 1 is the supertonic, 2 is the mediant, etc.*

Definition 3.9. *Let T_n^d be the diatonic transposition such that T_y^d on a diatonic pitch x gives $x + y\ (mod\ 7)$.*

[3]Our convention is that the diatonic pitch class is the scale degree minus one.

Definition 3.10. *Let* I_n^d *be the diatonic inversion such that* I_y^d *on a diatonic pitch* x *gives* $x - y \ (mod\ 7)$.

Example 3.11. $T_3^d(5) = 1$ *and* $I_1^d(6) = 2$.

Definition 3.12. *Let* $T^d := \{T_0^d, T_1^d, T_2^d, \ldots, T_6^d, I_0^d, I_1^d, I_2^d, \ldots, I_6^d\}$ *with the operation of function composition be the diatonic symmetry group, where* T_n^d *is the diatonic transposition by* n *and* I_n *is the diatonic inversion around* n.

Definition 3.13. *Let* $t_d : T^d \rightarrow Sym(P_d)$ *be the group action of diatonic transformation. This action takes in a diatonic transformation in* T^d *and expresses it as a permutation of the pitch classes in* P_d. *If* $T^d \subseteq Sym(P_d)$ *is a subgroup, then* $t_d : T \rightarrow Sym(P)$ *is an inclusion map with formula* $t_d(g) = g\ for\ g \in T^d$.

For any $p \in P_d$:

- The orbit of p under t_d, $Ord_{td}\ (p)$, is equal to P_d for all $\mathrm{p} \in P_d$ because every diatonic pitch q can be reached from p by $T_{q-p\ mod\ 7}^d$.
- The stabilizer of p, $Stab_{td}\ (p)$ is:

$$\{T_0^d, I_{2p\ mod\ 7}^d\}$$

because $p + 0 = p$ and $2p - p = p$.

When $x \neq 0$, the trajectory of T_x^d on *p* is equivalent to P_d because all integers from 1 to 6 are relatively prime to 7. The trajectory of T_0^d on p is $\{p\}$. For any x, the trajectory of I_0^d on p is:

$$\{p, x - p\}$$

because $x - (x - p) = x - x + p = p$.

3.2.1 Diatonic Symmetry Group and D_7 Are Isomorphic

Definition 3.14. *Let* $D_7 := \{O_0, O_1, O_2, \ldots, O_6, E_0, E_1, E_2, \ldots, E_6\}$ *be the dihedral group of degree 7. This group represents the rotational and reflectional symmetries of a regular heptagon (Figure 2).* O_n *represents a rotation by* n *points.*

- *If n is even, then E_n represents the symmetrical reflection over a line that passes through the point $n/2$.*

- *If n is odd, then E_n represents the reflection over a line that passes through the midpoint of the side connecting $\frac{n-1}{2}$ and $\frac{n+1}{2}$.*

Theorem 3.15. *T^d is isomorphic to D_7.*

Proof. D_7 has the presentation $\langle O_1, E_0 | ord(O_1) = 7, ord(E_0) = 2, E_0 O_1 E_0^{-1} = O_1^{-1} \rangle$.

T^d is generated by T_1^d and I_0^d. $ord(T_1^d) = 7$ and $ord(I_0^d) = 2$. $I_0^d T_1^d I_0^d$ on x is $0 - (0 - x + 1) = x - 1 \equiv x + 6 \ (mod\ 7)$ and $(T_1^d)^{-1} = T_6^d$. We have shown that T^d has the generators T_1^d and I_0^d and the relations $ord(T_1^d) = 7, ord(I_0^d) = 2, I_0^d T_1^d I_0^d = (T_1^d)^{-1}$.

The map $f : T^d \to D_7$ maps the generators and relations of T^d to the generators and relations of D_7, so f extends to a unique surjective group homomorphism. Both T^d and D_7 have 14 elements, so $|T^d| = |D_7|$.

Thus $T^d \cong D_7$ by Lemma 2.3.

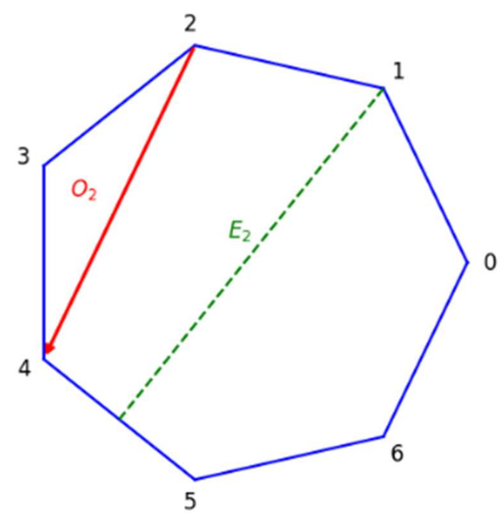

Figure 2: A heptagon with labeled points and example symmetries of D_7.

3.2.2 Is D_7 a good symmetry group for classical music?

D_7 does represent the diatonic scale well, and the diatonic scale is a major part of classical music, particularly the Baroque (1600–1750) and Classical (1750–1820) eras.

Another advantage is that to our ears, an increasing diatonic scale, for example, sounds like it goes up one by one, even if in 12-tone it is actually going up by 2, 2, 1, 2, etc. This means that we can think about the diatonic scale much more intuitively with the D_7 model than with the D_{12} model. An additional benefit is that the D_7 model provides us insight into classical music by removing the unneeded notes that were present in the D_{12} model and by allowing us to model music by focusing on scale degrees (e.g. tonic, dominant) instead of being restricted by the specific key that the music is in.

However, a few disadvantages come to mind. One obvious disadvantage is that there are definitely notes not in the key that are used in classical music, especially when modulating or using harmonic minor. Another drawback is that the D_7 model does not have a significant characteristic that defines the tonic. This means that major and natural minor, which both use the diatonic scale, cannot be distinguished, making this a problem because major and minor have very distinct sounds to the human ear. An additional downside is that the ratio of frequencies between diatonic intervals of the same size (such as between C and E and between D and F) are not the same. This is unlike the D_{12} model, where intervals that have the same size will always have the same frequency ratio (given equal temperament).

3.3 Diatonic Circle of Fifths

Given a chord, represented by a pitch class set, we can move it around the circle of fifths by repeatedly transposing it up by a perfect fifth.

Example 3.16. $T_7(\{0,4,7\}) = \{2,7,11\}$ *and* $T_7(\{2,7,11\}) = \{2,6,9\}$.

If we keep transposing the chord we will eventually have the same chord, thus forming a cycle known as the circle of fifths.

We want to make a diatonic version of this circle of fifths, but there is a problem. Consider the diatonic scale of C major. A perfect fifth up from B is $F\sharp$, which is not part of the C major scale. To get as close as possible to a fifth, we go from B to F natural instead.[4] Therefore our diatonic circle of fifths for C major is $C - G - D - A - E - B - F - C$.

[4]Going from B to G would result in C and F being skipped: B, G, D, A, E, B

3.3.1 T^d model

Figure 3: The diatonic circle of fifths as a series of ascending chords.

In our T^d model, the diatonic circle of fifths in general is:

$$\{0,2,4\},\{4,6,1\},\{1,3,5\},\{5,0,2\},\{2,4,6\},\{6,1,3\},\{3,5,0\},\{0,2,4\}$$

This is shown in Figure 3 as a series of ascending chords. We can observe that this is T_4^d repeatedly. In our geometric heptagon model, this is a rotation by 4 points. T_7 = T_4^d when the note that the transposition acts on is not scale degree 7.

In classical music, often a chord progression or phrase starts on the tonic, goes to the subdominant, and ends with the dominant and the tonic. We can obtain this progression with our circle of fifths, by going backwards in a ”circle of fourths”:

$$\{0,2,4\},\{3,5,0\},\{6,1,3\},\{2,4,6\},\{5,0,2\},\{1,3,5\},\{4,6,1\},\{0,2,4\}$$

This is shown in Figure 4 as a series of ascending chords. This would be T_3^d repeatedly.

Figure 4: The diatonic circle of fourths as a series of ascending chords.

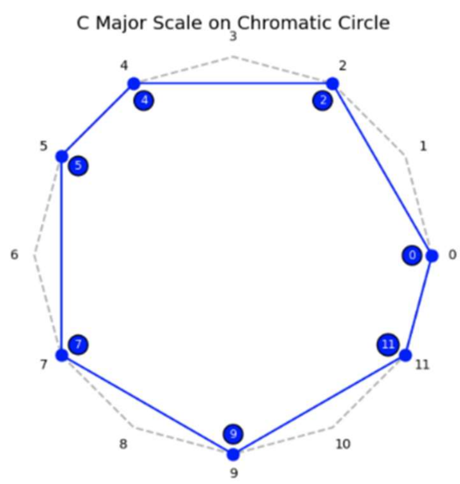

Figure 5: The diatonic scale represented geometrically on a dodecagon.

3.3.2 Connection between exception to $\boldsymbol{T_7 = T_4^d}$ and $\boldsymbol{I_4}$ symmetry of $\boldsymbol{D}$

In Section 3.1, it is mentioned that the diatonic pitch class set D has a nontrivial symmetry of I_4. There could be a connection between this symmetry and the one exception to $\boldsymbol{T_7 = T_4^d}$ mentioned in Section 3.3.1.

Let's represent D geometrically as the polygon formed by connecting 7 points of a dodecagon. (Figure 5)

First, if 7 points on a dodecagon that have distances between them of 1 or 2 have at least one reflectional symmetry, then is it true that at least one exception to $\boldsymbol{T_7 = T_4^d}$ exists? In other words, is it true that four adjacent edges of the heptagon have a distance (along the dodecagon) that is not 7? In fact, we do not even need the condition of having at least one reflectional symmetry.

Theorem 3.17. *If we are given 7 vertices of a dodecagon such that the maximum distance along the dodecagon between any two of them is 2, there will always be 2 vertices separated by 4 edges which have a distance which is not 7.*

Proof. The Pigeonhole Principle states that if n items are distributed among m containers and $n > m$, then at least one container must contain more than one item.[8] Consider that there are 6 pairs of points which have a distance of 6 between them (both clockwise and counterclockwise). Therefore, by the Pigeonhole Principle, there must be at least one pair of points that have a distance of 6 between them. Now we consider the ways in which edges of length 1 and 2 can cover a distance of 6.

1. six 1s: There would have to be an edge of length 6 to cover the remaining 6 distance, which is not allowed.

2. four 1s and one 2: Similarly, there would have to be two edges with an overall length of 6, which can't be done with edges of length 1 or 2.

3. two 1s and two 2s: Four edges cover a distance of 6, showing an exception to $\boldsymbol{T_4^d} = \boldsymbol{T_7}$.

4. three 2s: The remaining distance of 6 would have to be covered by the other four edges, showing an exception to $\boldsymbol{T_4^d} = \boldsymbol{T_7}$.

Thus, there must be an exception to $\boldsymbol{T_4^d} = \boldsymbol{T_7}$.

Now let us consider the other direction. If there are 7 points on a dodecagon that have distances between them of 1 or 2, and there is at least one exception to $\boldsymbol{T_7} = \boldsymbol{T_4^d}$, then is it true that they have at least one reflectional symmetry?

Theorem 3.18. *If we are given 7 vertices of a dodecagon such that the maximum distance along the dodecagon between any two of them is 2, and there are at least one pair of vertices that are separated by 4 edges which have a distance d which is not 7, then the heptagon formed by the 7 vertices must have at least one reflectional symmetry.*

Proof. We do casework on what d equals.

1. $d > 8$: The maximum distance 4 edges can cover is $4 \times 2 = 8$, so this is not possible.

2. $d = 8$: This can only be done by four 2s. Let's assume that their 5 vertices are $0, 2, 4, 6, 8$. The remaining 3 edges must then cover a distance of 4, so those 3 edges must be one 2 and two 1s. If we have those edges with vertices at:

   (a) 0, 11, 9, 8, then there is a symmetry across the line connecting 4 and 10.

   (b) 0, 10, 9, 8, then there is a symmetry across the line connecting 3 and 9.

   (c) 0, 11, 10, 8, then there is a symmetry similar to case b.

3. $d = 6$: The remaining 3 edges must then be three 2s to cover the distance of 6. Let's assume that their 4 vertices are $0, 2, 4, 6$. The 4 edges must then be two 2s and two 1s. If we have those edges with vertices at:

(a) 0, 11, 10, 8, 6: then there is a symmetry across the line connecting 5 and 11.

(b) 0, 11, 9, 8, 6: then there is a symmetry across the line connecting 4 and 10.

(c) 0, 11, 9, 7, 6: then there is a symmetry across the line connecting 3 and 9.

(d) 0, 10, 9, 8, 6: then there is a symmetry across the line connecting 3 and 9.

(e) 0, 10, 9, 7, 6: then there is a symmetry similar to case b.

(f) 0, 10, 8, 7, 6: then there is a symmetry similar to case a.

4. $d < 6$: The remaining 3 edges must then cover a distance greater than 6, which is not possible since $3 \times 2 = 6$.

3.3.3 12-tone T mode

In our T model, the diatonic circle of fifths in general is:

$$0, 7, 2, 9, 4, 11, 5, 0$$

Notice that, apart from 6 to 0, each interval is T_7. The interval from 6 to 0 is T_6, or a tritone.

Definition 3.19. *Let SC_p be the pentatonic set class and be comprised of the transpositions and inversions of* $\{0, 2, 4, 7, 9\}$.

Definition 3.20. *Let SC_t be the tritone set class and be comprised of the transpositions and inversions of* $\{5, 11\}$.

A diatonic fifth is then within either set class or a T_7 between the set classes.

Example 3.21. In C major, we start with 0 and then reach every pitch class in the pitch class set $\{0, 2, 4, 7, 9\}$ which is in SC_p: $0, 7, 2, 9, 4$. Then we take T_7 on 4 to get 11. Then we reach every pitch class in the pitch class set $\{11, 5\}$: $11, 5$. $\{11, 5\}$ is in SC_t. We then take T_7 on 5 to get 0.

Musically, the pentatonic scale is known to be very consonant (not dissonant) mainly because of its lack of tritones or minor seconds. The tritone is known as one of the two basic dissonant intervals. When it is added to the pentatonic scale, the diatonic scale is formed. The dissonant notes add minor seconds, which help to lead up to resolution. For example, in C major, F and B are the notes of the tritone, and F leads down to E and B leads up to C.

3.4 Triads

Definition 3.22. *The form of a chord is defined as the pitch class set which has the root note of the chord as 0 and the other notes as pitch classes relative to the root note.*

A major triad has the form $\{0, 4, 7\}$. A minor triad has the form $\{0, 3, 7\}$. $I_7(\{0, 4, 7\}) = \{0, 3, 7\}$ which is the form for a minor triad. Thus, a major triad is just the inverse of a minor triad.

A diminished triad (such as $B - D - F$) has the form $\{0, 3, 6\}$. $I_6(\{0, 3, 6\} = \{0, 3, 6\}$, so the inverse of a diminished triad is a diminished triad. If we have the diminished triad $dim = \{x, x + 3, x + 6\}$ which has root x, then $Stab_t(dim) = \{T_0, I_{2x+6}\}$.

An augmented triad (such as $C - E - G\#$) has the form $\{0, 4, 8\}$. Similar to the diminished triad, $I_8(\{0, 4, 8\} = \{0, 4, 8\}$, so the inverse of an augmented triad is an augmented triad.

Definition 3.23. *Let $aug := \{x, x + 4, x + 8\}$ be the augmented triad which has root x.*

$$Stab_t(aug) = \{T_0, T_4, T_8, I_{2x}, I_{2x+4}, I_{2x+8}\}.$$

Definition 3.24. *Let $T^{aug} := \{T_0, T_4, T_8, I_{2x}, I_{2x+4}, I_{2x+8}\}$ be the augmented triad symmetry group on aug.*

Definition 3.25. *Let $D_3 := \{O_0, O_1, O_2, E_0, E_1, E_2\}$ be the dihedral group of degree 3. This group represents the rotational and reflectional symmetries of an equilateral triangle. O_n represents a rotation by n points. If n is 0 or 2, then E_n represents the symmetrical reflection over a line that passes through the point $n/2$. E_1 represents the reflection over a line that passes through the midpoint of the side connecting 0 and 2.*

Theorem 3.26. *T^{aug} is isomorphic to D_3.*

Proof. D_3 has the presentation $\langle O_1, E_0 | ord(O_1) = 3, ord(E_0) = 2, E_0 O_1 E_0^{-1} = O_1^{-1}\rangle$.

T^{aug} is generated by T_4 and I_{2x}. $ord(T_4) = 3$ and $ord(I_{2x}) = 2$. $I_{2x} T_4 I_{2x}$ on aug is

$$\begin{aligned} &\{2x - (4 + (2x - x)), 2x - (4 + (2x - (x + 4))), 2x - (4 + (2x - (x + 8)))\} \\ &= \{2x - (4 + x), 2x - (4 + x - 4), 2x - (4 + x - 8)\} \\ &= \{x - 4, x, x + 4\} \\ &= \{x + 8, x + 12, x + 16\} \end{aligned}$$

which is T_8. $(T_4)^{-1} = T_8$. We have shown that T^{aug} has the generators T_4 and I_{2x} and the relations $ord(T_4) = 3$, $ord(I_{2x}) = 2$, $I_{2x} T_4 I_{2x} = (T_4)^{-1}$.

The map $f : T^{aug} \rightarrow D_3$ maps the generators and relations of T^{aug} to the generators and relations of D_3, so f extends to a unique surjective group homomorphism. Both T^{aug} and D_3 have 6 elements, so $|T^{aug}| = |D_3|$.

Thus $T^{aug} \cong D_3$ by Lemma 2.3.

3.4.1 Musical inversions of a generic triad

Let $\{a, b, c\}$ be a generic triad where a, b, and c are unique. Then the first musical inversion[5] of this triad is $\{b, c, a\}$ and the second inversion is $\{c, a, b\}$. If we forget the D_{12} structure, then the triad is now just three ordered points. The first inversion and the second inversion would then be two permutations of these points. Combining these with the identity permutation gives us the cyclic group of order 3, C_3, acting on $\{a, b, c\}$.

We can generalize this to a chord with any number of elements. Let $\{p_1, p_2, \ldots, p_n\}$ be a chord where $p_1, p_2, \ldots, p_n$ are unique. Then the i-th inversion of this chord is $p_{i+1}, p_{i+2}, \ldots, p_n, p_1, \ldots, p_i$. If we forget the D_{12} structure, then the chord is now just n ordered points. The inversions of the chord are then $n - 1$ inversions of these points. Combining these with the identity permutation gives us the cyclic group or order n, C_n, acting on $\{p_1, p_2, \ldots, p_n\}$.

3.5 Seventh Chords

3.5.1 Dominant seventh chord

A dominant seventh chord, such as $C, E, G, B\flat$, has the form $\{0, 4, 7, 10\}$. $I_7(\{0, 4, 7, 10\}) = \{0, 3, 7, 9\}$, or $C, E\flat, G, A$. This is the minor added sixth chord.

3.5.2 Major seventh chord

A major seventh chord, such as C, E, G, B, has the form $\{0, 4, 7, 11\}$. $I_{11}(\{0, 4, 7, 11\}) = \{0, 4, 7, 11\}$, so the inversion of a major seventh chord is a major seventh chord. If we have the major seventh chord $M7 = \{x, x + 4, x + 7, x + 11\}$, then $Stab_t(M7) = \{T_0, I_{2x+11}\}$.

3.5.3 Minor seventh chord

A minor seventh chord, such as $C, E\flat, G, B\flat$, has the form $\{0, 3, 7, 10\}$. $I_{10}(\{0, 3, 7, 10\} = \{0, 3, 7, 10\}$, so the inversion of a minor seventh chord is a minor seventh chord. If we have the minor seventh chord $m7 = \{x, x + 3, x + 7, x + 10\}$, then $Stab_t(m7) = \{T_0, I_{2x+1}\}$.

[5] Not to be confused with I_n, this refers to a rearrangement of notes such that the root note changes

3.5.4 Diminished seventh chord

A diminished seventh chord (such as $C, E\flat, F\sharp, A$) has the form $\{0, 3, 6, 9\}$. $I_9(\{0, 3, 6, 9\}) = \{0, 3, 6, 9\}$, so the inverse of a diminished seventh chord is a diminished seventh chord.

Definition 3.27. *Let $dim7 := \{x, x + 3, x + 6, x + 9\}$ be the diminished 7th chord which has root x.*

$$Stab_t(dim7) = \{T_0, T_3, T_6, T_9, I_{2x}, I_{2x+3}, I_{2x+6}, I_{2x+9}\}.$$

Definition 3.28. *Let $T^{dim7} := \{T_0, T_3, T_6, T_9, I_{2x}, I_{2x+3}, I_{2x+6}, I_{2x+9}\}$ be the diminished 7th chord symmetry group on $dim7$.*

Definition 3.29. *Let $D_4 := \{O_0, O_1, O_2, O_3, E_0, E_1, E_2, E_3\}$ be the dihedral group of degree 4. This group represents the rotational and reflectional symmetries of a square. O_n represents a rotation by n points. If n is 0 or 2, then E_n represents the symmetrical reflection over a line that passes through the point $n/2$. If n is 1 or 3, E_n represents the reflection over a line that passes through the midpoint of the side connecting $\frac{n-1}{2}$ and $\frac{n+1}{2}$.*

Theorem 3.30. *T^{dim7} is isomorphic to D_4.*

Proof. D_4 has the presentation $\langle O_1, E_0 | ord(O_1) = 4, ord(E_0) = 2, E_0 O_1 E_0^{-1} = O_1^{-1} \rangle$.

T^{dim7} is generated by T_3 and I_{2x}. $ord(T_3) = 4$ and $ord(I_{2x}) = 2$. $I_{2x}T_3I_{2x}$ on $dim7$ is

$$\begin{aligned} &\{2x - (3 + (2x - x)), 2x - (3 + (2x - (x + 3))), \\ &2x - (3 + (2x - (x + 6))), 2x - (3 + (2x - (x + 9)))\} \\ &= \{2x - (3 + x), 2x - (3 + x - 3), 2x - (3 + x - 6), 2x - (3 + x - 9)\} \\ &= \{x - 3, x, x + 3, x + 6\} \\ &= \{x + 9, x + 12, x + 15, x + 18\} \end{aligned}$$

which is T_9. $(T_3)^{-1} = T_9$. We have shown that T^{dim7} has the generators T_3 and I_{2x} and the relations $ord(T_3) = 4$, $ord(I_{2x}) = 2$, $I_{2x}T_3I_{2x} = (T_3)^{-1}$.

The map $f : T^{dim7} \rightarrow D_4$ maps the generators and relations of T^{dim7} to the generators and relations of D_4, so f extends to a unique surjective group homomorphism. Both T^{dim7} and D_4 have 8 elements, so $|T^{dim7}| = |D_4|$.

Thus $T^{dim7} \cong D_4$ by Lemma 2.3.

3.5.5 Example of using the symmetries of diminished 7th chords

In Chopin's ´Etude in $C\sharp$minor, Op. 10, No. 4, there is a descending pattern of diminished 7th chords, as shown in the third and fourth measures of Figure 6.

Figure 6: An example of using diminished seventh chords in Chopin's Étude in $C\sharp$ minor, Op. 10, No. 4.

In the left hand, the pattern starts on the second beat with $G\sharp, D, B, E\sharp$. This is a sequence where first T_6 acts on $G\sharp$: $T_6(8) = 2$ which is D. Then T_9 acts on D: $T_9(2) = 11$ which is B. Then T_6 acts on B: $T_6(11) = 5$ which is $F = E\sharp$. The same sequence is then repeated in the next beat but down 3 pitches, or equally up 9 pitches and down an octave. This is essentially a T_9 of the four-note sequence. After four of these four-note sequences, we observe that the notes repeat but down an octave. We can also observe that all of the notes in this pattern are always one of four notes $D, E\sharp, G\sharp, B$. Why are these observations true? Well, the four pitches form a diminished 7th chord, and all of the transpositions are elements of T^{dim7}, so any other pitches formed by these transpositions acting on the four pitches will have to be one of those notes. Additionally, since $T^{dim7} \cong D_4$, we can think of these four notes as the four corners of a square. Then, the transpositions from one note to the next are rotational symmetries of the square.

3.6 Whole tone scale

A whole tone scale (such as $C, D, E, F\sharp, G\sharp, A\sharp, (C))$ has the form $\{0, 2, 4, 6, 8, 10\}$.

Definition 3.31. *Let* $wt := \{x, x+2, x+4, x+6, x+8, x+10\}$ *be the whole tone scale which has root* x.

$$Stab_t(wt) = \{T_0, T_2, T_4, T_6, T_8, T_{10}, I_{2x}, I_{2x+2}, I_{2x+4}, I_{2x+6}, I_{2x+8}, I_{2x+10}\}.$$

Definition 3.32. *Let* $T^{wt} := \{T_0, T_2, T_4, T_6, T_8, T_{10}, I_{2x}, I_{2x+2}, I_{2x+4}, I_{2x+6}, I_{2x+8}, I_{2x+10}\}$ *be the whole tone scale symmetry group on* wt.

Definition 3.33. *Let* $D_6 := \{O_0, O_1, O_2, O_3, O_4, O_5, E_0, E_1, E_2, E_3, E_4, E_5\}$ *be the dihedral group of degree 6. This group represents the rotational and reflectional symmetries of a regular hexagon.* O_n *represents a rotation by* n *points. If* n *is even, then* E_n *represents the symmetrical reflection over a line that passes through the point* $n/2$. *If* n *is odd,* E_n *represents the reflection over a line that passes through the midpoint of the side connecting* $\frac{n-1}{2}$ *and* $\frac{n+1}{2}$.

Theorem 3.34. T^{wt} *is isomorphic to* D_6.

Proof. D_6 has the presentation $\langle O_1, E_0 | ord(O_1) = 6, ord(E_0) = 2, E_0 O_1 E_0^{-1} = O_1^{-1} \rangle$.

T^{wt} is generated by T_2 and I_{2x}. $ord(T_2) = 6$ and $ord(I_{2x}) = 2$. $I_{2x}T_3I_{2x}$ on wt is

$$\begin{aligned}
&\{2x - (2 + (2x - x)), 2x - (2 + (2x - (x + 2))), 2x - (2 + (2x - (x + 4))),\\
&2x - (2 + (2x - (x + 6))), 2x - (2 + (2x - (x + 8))), 2x - (2 + (2x - (x + 10)))\}\\
&= \{2x - (2 + x), 2x - (2 + x - 2), 2x - (2 + x - 4),\\
&2x - (2 + x - 6), 2x - (2 + x - 8), 2x - (2 + x - 10)\}\\
&= \{x - 2, x, x + 2, x + 4, x + 6, x + 8\}\\
&= \{x + 10, x + 12, x + 14, x + 16, x + 18, x + 20\}
\end{aligned}$$

which is T_{10}. $(T_2)^{-1} = T_{10}$. We have shown that T^{wt} has the generators T_2 and I_{2x} and the relations $ord(T_2) = 6$, $ord(I_{2x}) = 2$, $I_{2x}T_2I_{2x} = (T_2)^{-1}$.

The map $f : T^{wt} \to D_6$ maps the generators and relations of T^{wt} to the generators and relations of D_6, so f extends to a unique surjective group homomorphism. Both T^{wt} and D_6 have 12 elements, so $|T^{wt}| = |D_6|$.

Thus $T^{wt} \cong D_6$ by Lemma 2.3.

3.7 Pentatonic scale

A pentatonic scale (such as $C, D, E, G, A, (C)$), has the form $\{0, 2, 4, 7, 9\}$. $I_4(\{0, 2, 4, 7, 9\}) = \{0, 2, 4, 7, 9\}$, so the inversion of a pentatonic scale is a pentatonic scale. If we have the pentatonic scale $pt = \{x, x + 2, x + 4, x + 7, x + 9\}$, then $Stab_t(pt) = \{T_0, I_{2x+4}\}$.

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