A Geometric Exploration of Generalized Mandelbrot Sets

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1 Introduction

The Mandelbrot set is the set of complex numbers that remain bounded after taking iterates of the polynomial $f(z) = z^2 + c$, generating a sequence. It is a beautiful figure that illustrates the intricate relationship between complex numbers and fractal geometry.

The Mandelbrot set was first discovered by Benoit Mandelbrot in the late 1970s. During this era, mathematicians were trying to find ways on how computers could complement their research, resulting in areas of interest such as data visualization. Mandelbrot was working on a phone line transmission problem, analyzing a plot of failures over time. Coincidentally, as he was working on this problem, he was reminded of the Julia Set, discovered by Gaston Julia in 1918. This reminder led Mandelbrot to the notion of fractals. Fractals are self-replicating figures that repeat on infinitely many levels, giving rise to beautiful figures. Today, real world applications of fractals include movie and game animators, apartment designs, and the context of nature. Mandelbrot's findings are extensively documented in The Fractal Geometry of Nature [3]. His work was a basis for more advanced research in fractal geometry. Despite numerous computational estimates, an exact expression for the area of the Mandelbrot set still does not exist, as discussed in [2]. Subsequent research can be found from sources such as the work in [1], which attempts to give estimates using complex geometry and computation.

Our research extends the study of Mandelbrot sets by analyzing the generalization of $f(z) = z^2 + c$ to $f(z) = z^j + c$. By plotting these generalized Mandelbrot sets using computational simulations, we generalize trends regarding the Mandelbrot set area approximations. Additionally, we explore their structural properties, including their rotational and reflectional symmetries. We record and prove these key symmetries using explicit algebraic formulae. This contributes to the literature by making these symmetries understandable using non-group theoretic ideas.

The paper begins by establishing definitions and preliminaries necessary for understanding how the Mandelbrot set works. The next section highlights a formal proof of the escape criterion, followed by the escape-time algorithm and numerical area approximations. Conjectures regarding generalized trends for the area and escape radius, and symmetry are made based on computational evidence. Finally, lines of symmetry are derived, and rotational and reflectional symmetries are proved. The paper concludes with some unique and interesting petal-like properties exhibited by the generalized Mandelbrot set.

2 Definitions

Definition 1 *A complex number c is bounded by a real number R if* $|c| < R$*.*

Definition 2 *The jth generalized Mandelbrot set is the set of points c in the complex plane such that the nth iterates of* $f(z) = z^j + c$ *is bounded by some number R, independent of n for all n.*

Definition 3 *Consider a polynomial of the form* $f(z) = z^j + c$*, where j is a natural number and c is a complex number. The nth iterate is the result of applying the polynomial f to the input z=0 n times, i.e.* $f(f(f(...0)))$ *where there are n functions in the composition. For clarity, the 0th iterate is just 0.*

The polynomial iterates over z_n starting with 0, creating a sequence of complex numbers that lie in the Mandelbrot set, with the use of a recursive formula. As an example, let's analyze $f(z) = z^2 + c$.

$f(0) = 0^2 + c = c$
$f(c) = c^2 + c$
$f(c^2 + c) = (c^2 + c)^2 + c$
$f((c^2 + c)^2 + c) = ((c^2 + c)^2 + c)^2 + c$
...

Observe that just after a few iterations of z_n, the behavior of $f(z)$ becomes increasingly complex. To study the generalized Mandelbrot set more efficiently, we analyze its escape criterion, which states that the Mandelbrot set is bounded by a ball of radius 2 (we will prove this in the next section).

3 Escape Criterion Proof for Generalized Mandelbrot Sets

To approximate the generalized Mandelbrot sets, we need a simple, computable criterion for when a point is not in the set, which is known as the escape criterion. We model the escape criterion proof given in [6] for specific values of j, and then generalize the proof for all j.

Lemma 3.1 *If c is in the generalized Mandelbrot set, then c is bounded by a norm of 2.*

Proof of Lemma 3.1

Proof for j = 2

Assume $|z_n| = 2 + a, a > 0 \ (|z_n| > 2)$
$f(z) = z^2 + c \rightarrow z_{n+1} = z_n^2 + c$

1) Let $|c| \leq 2$:
$|z_{n+1}| = |z_n^2 + c| \geq |z_n|^2 - |c| = (2+a)^2 - 2 = 2 + 2a + a^2 > 2 + 2a$ (triangle inequality)
By induction, $|z_{n+k}| > 2 + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.

2) Let $|c| > 2$:
For $n = 0:$ $|z_{n+1}|$ = $|z_{0+1}|$ = $|z_0^2 + c|$ = $0^2 + c$ = $|c|$
$\rightarrow |z_1| > 2$, which already guarantees divergence.

Proof for j = 3

Assume $|z_n| = 2 + a, a > 0 \ (|z_n| > 2)$
$f(z) = z^3 + c \rightarrow z_{n+1} = z_n^3 + c$
As proved earlier, for $|c| > 2$, divergence is guaranteed, so we will proceed for $|c| \leq 2$.
$|z_{n+1}| = |z_n^3 + c| \geq |z_n|^3 - |c| = (2+a)^3 - 2 = 6 + 12a + 6a^2 + a^3 > 6 + 12a$ (triangle inequality)
By induction, $|z_{n+k}| > 6 + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.
.
.
.
Proof for j = 1000

Assume $|z_n| = 2 + a, a > 0 \ (|z_n| > 2)$
$f(z) = z^{1000} + c \rightarrow z_{n+1} = z_n^{1000} + c$
$|z_{n+1}| = |z_n^{1000} + c| \geq |z_n|^{1000} - |c| = (2+a)^{1000} - 2$ (triangle inequality)
$(2+a)^{1000} = 2^{1000} + 1000(2^{999})(a)$ + ...

$\rightarrow (2+a)^{1000} \geq 2^{1000} + 1000(2^{999})(a)$
$\rightarrow (2+a)^{1000} - 2 \geq 2^{1000} + 1000(2^{999})(a) - 2$
$\rightarrow |z_{n+1}| \geq 2^{1000} + 1000(2^{999})(a) - 2$
By induction, $|z_{n+k}| > 2^{1000} + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.

Proof for General j

Assume $|z_n| = 2 + a, a > 0$ $(|z_n| > 2)$
$f(z) = z^j + c \rightarrow z_{n+1} = z_n^j + c$
$|z_{n+1}| = |z_n^j + c| \geq |z_n|^j - |c| = (2+a)^j - 2$ (triangle inequality)
$(2+a)^j = 2^j + j(2^{j-1})(a)$ + ...
$\rightarrow (2+a)^j \geq 2^j + j(2^{j-1})(a)$
$\rightarrow (2+a)^j - 2 \geq 2^j + j(2^{j-1})(a) - 2$
$\rightarrow |z_{n+1}| \geq 2^j + j(2^{j-1})(a) - 2$
By induction, $|z_{n+k}| > 2^j + ak \rightarrow \infty$ as $k \rightarrow \infty$
$\rightarrow |z_{n+1}| > |z_n|$
$\therefore z_n$ diverges to infinity if $|z_n| > 2$.

$\therefore$ By induction, we conclude that the generalized Mandelbrot set $f(z) = z^j + c$ lies in a ball of radius 2.

4 How do we measure its area?

Unlike its infinite perimeter, the Mandelbrot set has a finite area. To this day, the area of the Mandelbrot set to mathematicians is unknown! Its estimated area is 1.506484 square units. By running simulations, approximations of the area can be made with the escape-time algorithm.

5 The Escape Time Algorithm

While we technically cannot check if the iterates are always bounded, we can immediately see if it ever leaves the ball of radius two and is thus not in the (generalized) Mandelbrot set. Most modern implementations are based on [3], including [7], which is written in Python. Our implementation is specifically based on [7]. It is implemented as follows :

1) Generate a complex matrix (the complex plane) using numpy vectorization.

2) Start with $z_0 = 0$, then iterate over $f(z) = z^j + c$ using recursion.

3) The last step is to check (true or false) whether the point escapes the ball of radius 2 within the number of iterations set in the algorithm.

+ Code + Text

[1]
```
import matplotlib.pyplot as plt
import numpy as np
import warnings

warnings.filterwarnings("ignore")
```

[2]
```
def complex_matrix(xmin, xmax, ymin, ymax, pixel_density):
    re = np.linspace(xmin, xmax, int((xmax - xmin) * pixel_density))
    im = np.linspace(ymin, ymax, int((ymax - ymin) * pixel_density))
    return re[np.newaxis, :] + im[:, np.newaxis] * 1j
```

[38]
```
def is_stable(c, num_iterations):
    z = 0
    for _ in range(num_iterations):
        z = z**17+c
    return abs(z) <= 2
```

[39]
```
def get_members(c, num_iterations):
    mask = is_stable(c, num_iterations)
    return c[mask]
```

[40]
```
if __name__ == "__main__":
    c = complex_matrix(-2, 2, -2, 2, pixel_density=512)
    members = get_members(c, num_iterations=25)

    plt.scatter(members.real, members.imag, color="black", marker=",", s=1)
    plt.gca().set_aspect("equal")
    plt.axis("on")
    plt.tight_layout()
    plt.show()
```

By finding the proportion of the members of the Mandelbrot set to the total number of pixels in the complex matrix, then multiplying by the complex matrix dimensions, we obtain the set's area.

6 Interesting Conjectures

The following conjectures are derived by running the escape-time algorithm for generalized Mandelbrot sets, for several values of j. For each value of j, we compute its corresponding area approximation. To ensure accuracy of the area approximations, we ran the algorithm for different parameter values, specifically the number of iterations and pixel density. By increasing the number of iterations and pixel density, the area approximation becomes more accurate. Accuracy is generally expected to increase as the number of iterations increases, because the algorithm has more time to determine whether the points belong to the Mandelbrot set. Similarily, increasing the pixel density improves the resolution of the set, reducing sampling errors. After a certain number of iterations, approximations stabilize, indicating consistency, which lets us know when to stop running the algorithm for an area approximation. We average the stabilized approximations, and record them as our area. A table is created, representing coordinates of the form (j, area). From there, we plot these coordinates on Desmos, and calculate the best curve that generalizes the j vs area trend. The tables of values specific to each iteration is included at the end of the paper, in Section 11. We keep the pixel density constant, as it does not have a major impact on the area approximation.

j	Area
3	1.8
4	1.98
5	2.115
6	2.2175
7	2.2972
8	2.3625
9	2.4169
10	2.4628
11	2.5021
12	2.5364
13	2.5674
14	2.5942
15	2.6184
16	2.6410
17	2.6605
18	2.6793
19	2.6960
20	2.7115
50	2.919038772583008
80	2.9850730896
200	3.0645503998

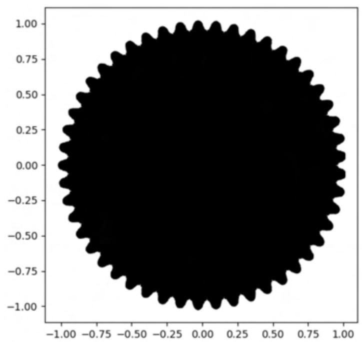
Table 1: Comparison of j values and their respective areas.

Conjecture 1 *The degree, j, and the corresponding area approximation of its polynomial form a general trend, fitting a curve.*

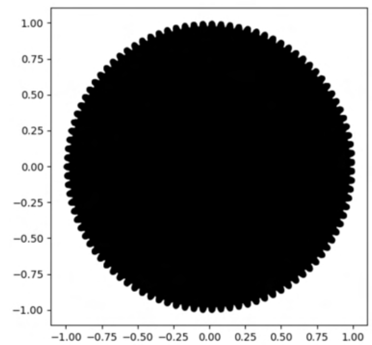
The best-fit curve based on my table of values is:

$$C(j) = \pi - \frac{2.6057}{j^{0.5264}}$$

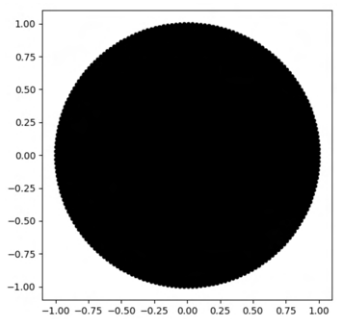
Conjecture 2 *For extremely large values of j, the Mandelbrot set is bounded by the unit disk.*



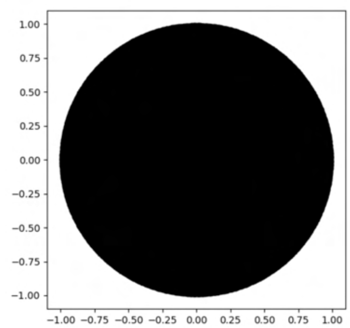
$f(z) = z^{50} + c$



$f(z) = z^{100} + c$



$f(z) = z^{200} + c$



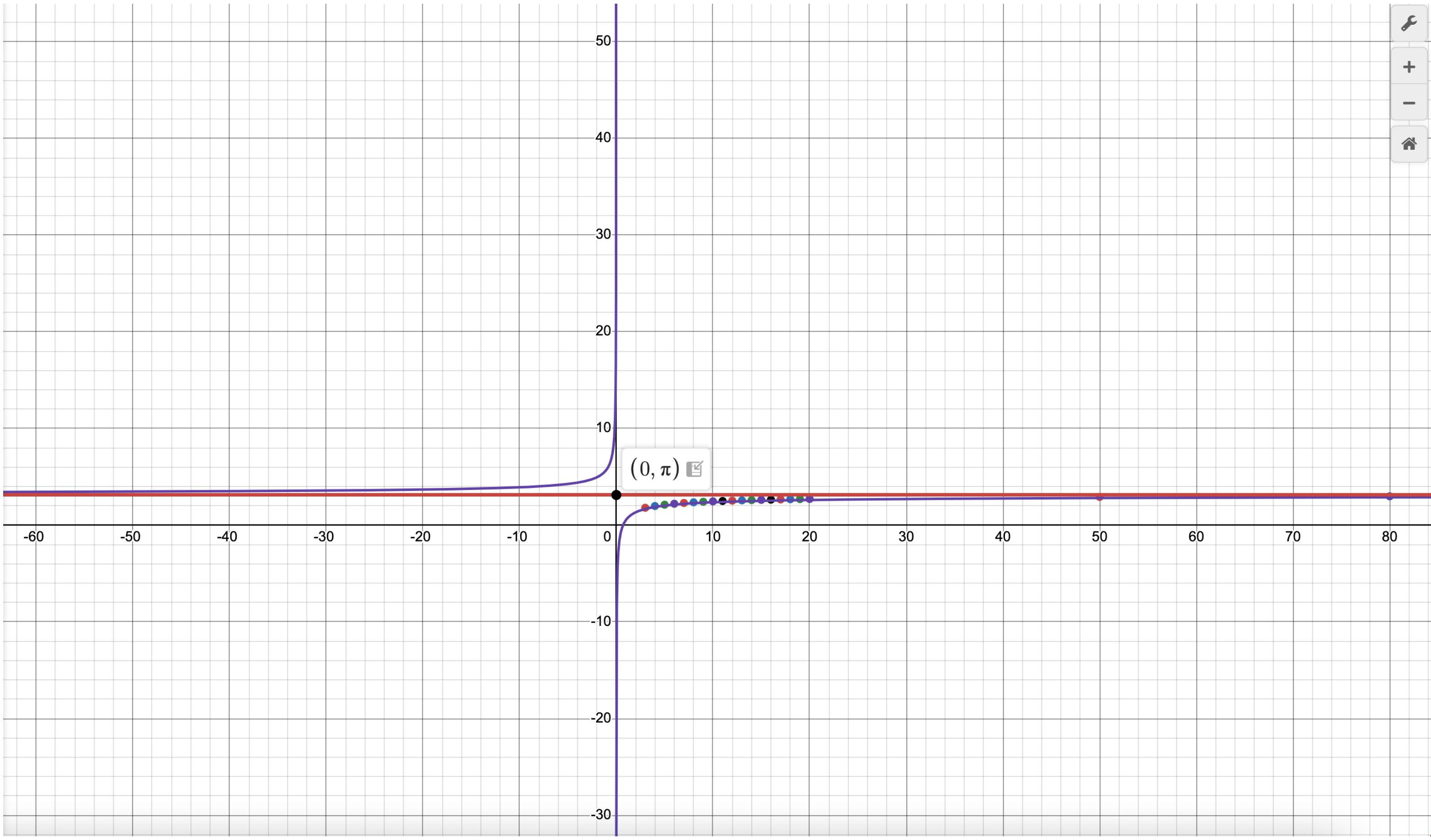
$f(z) = z^{1000} + c$

Observe that for smaller values of j, the boundary of the set is jagged and exhibits a gear-like structure. As j increases significantly, the curvature of the boundary becomes more uniform.

Conjecture 3 *As j increases, the radius of the ball decreases.*

This is a consequence of our previous conjecture. We initially proved the escape criterion in a previous section, which states that the Mandelbrot set is contained in a disk of radius 2. However, our results from our previous conjecture show that for significantly large values of j, the Mandelbrot set is bounded by the unit disk. This suggests that as j increases, the radius decreases.

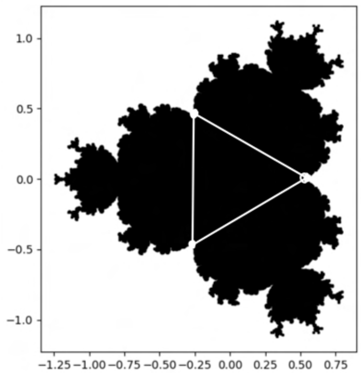
Conjecture 4 *As j approaches infinity, the area approaches π.*



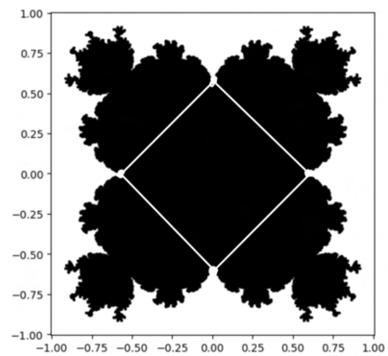
This conjecture is reinforced by Conjecture 2, where we found the Mandelbrot set to be bounded by the unit disk for extremely large values of j. We know that the area of the unit disk is π. By graphing our curve and the line $y = \pi$, we can visually observe the asymptotic behavior at π, and as $j \to \infty$, area $\to \pi$.

7 Symmetries

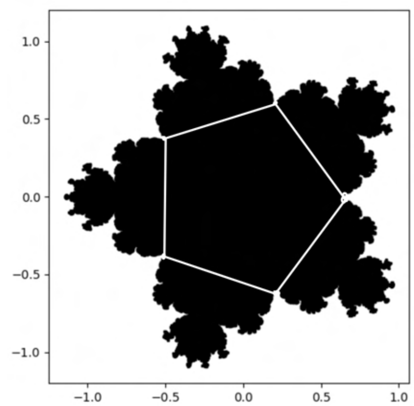
We will now explore and prove various reflectional and rotational symmetries of the Mandelbrot set. Lets closely analyze the 4th, 5th, and 6th Mandelbrot sets.



$f(z) = z^4 + c$
3-gon



$f(z) = z^5 + c$
4-gon

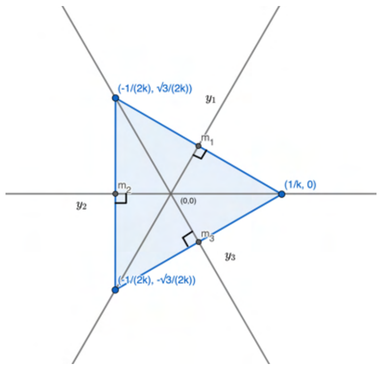


$f(z) = z^6 + c$
5-gon

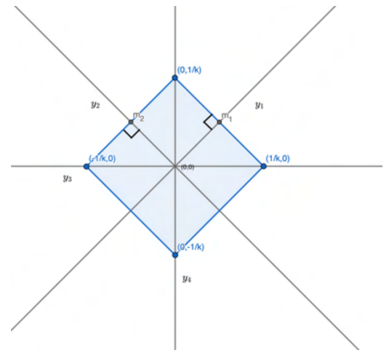
The roots of unity are elegantly inscribed in the Mandelbrot sets. More specifically, for a set of degree j, a j-1 - gon is inscribed in it, containing j-1 roots of unity as its vertices. However, when taking a closer look, we can see that the roots of unity are dilated, forming smaller j-1 - gons. Let us set that dilation factor to k, and derive the equations of these lines in the next section.

7.1 Deriving Lines of Symmetries

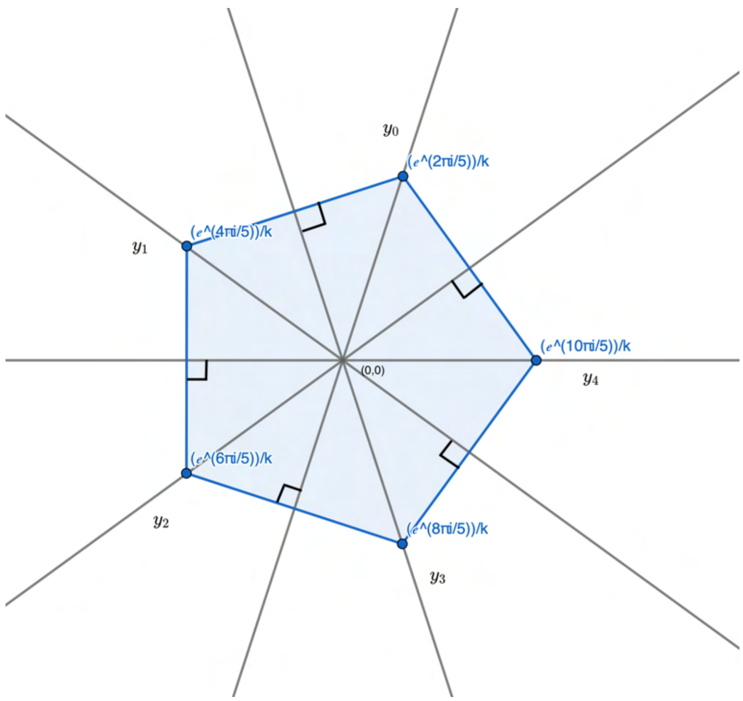
We begin by solving for the roots of unity, and converting them to cartesian coordinates. Next, we assign them as vertices of our $j-1$ - gons and draw perpendicular bisectors from the vertices to their opposite sides, passing through the origin. From there, derivation becomes trivial by nature when applying the slope-intercept form for the origin and each of the vertices.



$f(z) = z^4 + c$
Cube roots of unity



$f(z) = z^5 + c$
4th roots of unity



$f(z) = z^6 + c$
5th roots of unity

$\underline{f(z) = z^4 + c}$

$(kz)^3 - 1 = 0$

$(kz)^3 = 1 = e^{2\pi i}$

$kz_0 = 1$
$kz_1 = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
$kz_2 = e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
$\rightarrow$
$z_0 = \frac{1}{k}; (\frac{1}{k}, 0)$
$z_1 = \frac{1}{k}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i); (-\frac{1}{2k}, \frac{\sqrt{3}}{2k})$
$z_2 = \frac{1}{k}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i); (-\frac{1}{2k}, -\frac{\sqrt{3}}{2k})$

$y_1 : (0,0), (-\frac{1}{2k}, -\frac{\sqrt{3}}{2k})$
$m = \frac{-\frac{\sqrt{3}}{2k}-0}{-\frac{1}{2k}-0} = \sqrt{3}$
$y_1 = \sqrt{3}x_1$

$y_2 : (0,0), (\frac{1}{k}, 0)$
$m = \frac{0-0}{\frac{1}{k}-0} = 0$
$y_2 = 0$

$y_3 : (0,0), (-\frac{1}{2k}, \frac{\sqrt{3}}{2k})$
$m = \frac{\frac{\sqrt{3}}{2k}-0}{-\frac{1}{2k}-0} = -\sqrt{3}$
$y_3 = -\sqrt{3}x_3$

$\underline{f(z) = z^5 + c}$

$(kz)^4 - 1 = 0$
$(kz)^4 = 1 = e^{2\pi i}$

$kz_0 = e^{2\pi i/4} = i$
$kz_1 = e^{4\pi i/4} = -1$
$kz_2 = e^{4\pi i/4} = -i$
$kz_3 = e^{8\pi i/4} = 1$
$\rightarrow$
$z_0 = \frac{i}{k}; (0, \frac{1}{k})$
$z_1 = -\frac{1}{k}; (-\frac{1}{k}, 0)$
$z_2 = -\frac{i}{k}; (0, -\frac{1}{k})$
$z_3 = \frac{1}{k}; (\frac{1}{k}, 0)$

$m_1 : (0, \frac{1}{k}), (\frac{1}{k}, 0)$
$\frac{0+\frac{1}{k}}{2} = \frac{1}{2k}, \frac{\frac{1}{k}+0}{2} = \frac{1}{2k}; (\frac{1}{2k}, \frac{1}{2k})$
$y_1 : (\frac{1}{2k}, \frac{1}{2k}), (0,0)$

$m = \frac{0-\frac{1}{2k}}{0-\frac{1}{2k}} = 1$
$y_1 = x_1$

$m_2 : (0, \frac{1}{k}), (-\frac{1}{k}, 0)$
$\frac{0+(-\frac{1}{k})}{2} = -\frac{1}{2k}, \frac{\frac{1}{k}+0}{2} = \frac{1}{2k}; (-\frac{1}{2k}, \frac{1}{2k})$
$y_2 : (-\frac{1}{2k}, \frac{1}{2k}), (0, 0)$
$m = \frac{0-\frac{1}{2k}}{0-(-\frac{1}{2k})} = -1$
$y_2 = -x_2$

$y_3 : (-\frac{1}{k}, 0), (\frac{1}{k}, 0)$
$m = \frac{0-0}{\frac{1}{k}-(-\frac{1}{k})} = 0$
$y_3 = 0$

$y_4 : (0, \frac{1}{k}), (0, -\frac{1}{k})$
$m = \frac{-\frac{1}{k}-\frac{1}{k}}{0-0}$ DNE
$x = 0$

$\underline{f(z) = z^6 + c}$

$(kz)^5 - 1 = 0$
$(kz)^5 = 1 = e^{2\pi i}$

$kz_0 = e^{2\pi i/5}; (\cos(\frac{2\pi}{5}), \sin(\frac{2\pi}{5}))$
$kz_1 = e^{4\pi i/5}; (\cos(\frac{4\pi}{5}), \sin(\frac{4\pi}{5}))$
$kz_2 = e^{4\pi i/5}; (\cos(\frac{6\pi}{5}), \sin(\frac{6\pi}{5}))$
$kz_3 = e^{8\pi i/5}; (\cos(\frac{8\pi}{5}), \sin(\frac{8\pi}{5}))$
$kz_3 = e^{10\pi i/5}; (\cos(\frac{10\pi}{5}), \sin(\frac{10\pi}{5}))$
$\rightarrow$
$z_0 : (\frac{\cos(\frac{2\pi}{5})}{k}, \frac{\sin(\frac{2\pi}{5})}{k})$
$z_1 : (\frac{\cos(\frac{4\pi}{5})}{k}, \frac{\sin(\frac{4\pi}{5})}{k})$
$z_2 : (\frac{\cos(\frac{6\pi}{5})}{k}, \frac{\sin(\frac{6\pi}{5})}{k})$
$z_3 : (\frac{\cos(\frac{8\pi}{5})}{k}, \frac{\sin(\frac{8\pi}{5})}{k})$
$z_4 : (\frac{\cos(\frac{10\pi}{5})}{k}, \frac{\sin(\frac{10\pi}{5})}{k})$

$y_0 : (\frac{\cos(\frac{2\pi}{5})}{k}, \frac{\sin(\frac{2\pi}{5})}{k}), (0, 0)$
$m = \frac{0-\frac{\sin(\frac{2\pi}{5})}{k}}{0-\frac{\cos(\frac{2\pi}{5})}{k}} = \tan(\frac{2\pi}{5})$
$y_0 = \tan(\frac{2\pi}{5})x_0$

$$y_1 : (\frac{\cos(\frac{4\pi}{5})}{k}, \frac{\sin(\frac{4\pi}{5})}{k}), (0,0)$$
$$m = \frac{0 - \frac{\sin(\frac{4\pi}{5})}{k})}{0 - \frac{\cos(\frac{4\pi}{5})}{k}} = \tan(\frac{4\pi}{5})$$
$$y_1 = \tan(\frac{4\pi}{5})x_1$$

$$y_2 : (\frac{\cos(\frac{6\pi}{5})}{k}, \frac{\sin(\frac{6\pi}{5})}{k}), (0,0)$$
$$m = \frac{0 - \frac{\sin(\frac{6\pi}{5})}{k})}{0 - \frac{\cos(\frac{6\pi}{5})}{k}} = \tan(\frac{6\pi}{5})$$
$$y_2 = \tan(\frac{6\pi}{5})x_2$$

$$y_3 : (\frac{\cos(\frac{8\pi}{5})}{k}, \frac{\sin(\frac{8\pi}{5})}{k}), (0,0)$$
$$m = \frac{0 - \frac{\sin(\frac{8\pi}{5})}{k})}{0 - \frac{\cos(\frac{8\pi}{5})}{k}} = \tan(\frac{8\pi}{5})$$
$$y_3 = \tan(\frac{8\pi}{5})x_3$$

$$y_4 : (\frac{\cos(\frac{10\pi}{5})}{k}, \frac{\sin(\frac{10\pi}{5})}{k}), (0,0)$$
$$m = \frac{0 - \frac{\sin(\frac{10\pi}{5})}{k})}{0 - \frac{\cos(\frac{10\pi}{5})}{k}} = \tan(\frac{10\pi}{5})$$
$$y_4 = \tan(\frac{10\pi}{5})x_4$$

Remark : Notice that the lines of symmetry we derived are exactly the same as those passing through the roots of unity, regardless of any dilation. Since these lines pass through the origin, their angular positions remain unchanged under dilation, preserving the symmetry structure.

The following three technical lemmas are the basic ingredients for proving Theorem 7.4, our symmetry theorem regarding reflectional and rotational symmetries.

We use the well-known reflectional symmetry formula, m being the slope, where $R(x, y) = (\frac{(1-m^2)x+2my}{1+m^2}, \frac{(m^2-1)y+2mx}{1+m^2})$
This can be proven using linear algebra, as shown in [8].

Let us define $r(c)$ to be the rotational symmetry formula, where $r(c) = e^{\frac{2(\pi)ik}{j-1}} * c$

Lemma 7.1 *Each of the pairs j and R listed below satisfies $|R(z)| = |z|$ (follows for j and r)*

Case 1 : $j = 4$ and $R(x, y) = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x + \frac{1}{2}y)$
Case 2 : $j = 4$ and $R(x, y) = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$
Case 3 : $j = 4$ and $R(x, y) = (x, -y)$
Case 4 : $j = 4$ and $r(c) = c * e^{2\pi ik/3}, k = 0, 1, 2$
Case 5 : $j = 5$ and $R(x, y) = (y, x)$
Case 6 : $j = 5$ and $R(x, y) = (x, -y)$
Case 7 : $j = 5$ and $R(x, y) = (-y, -x)$
Case 8 : $j = 5$ and $R(x, y) = (-x, y)$
Case 9 : $j = 5$ and $r(c) = c * e^{2\pi ik/4}, k = 0, 1, 2, 3$
Case 10 : $j = 6$ and $R(x, y) = (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{\sec^2(\frac{2\pi}{5})})$
Case 11 : $j = 6$ and $r(c) = c * e^{2\pi ik/5}, k = 0, 1, 2, 3, 4$

Lemma 7.2 *Each of the pairs j and R listed below satisfies $R(z^j) = (R(z))^j$ (follows for j and r)*

Case 1 : $j = 4$ and $R(x, y) = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x + \frac{1}{2}y)$
Case 2 : $j = 4$ and $R(x, y) = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$
Case 3 : $j = 4$ and $R(x, y) = (x, -y)$
Case 4 : $j = 4$ and $r(c) = c * e^{2\pi ik/3}, k = 0, 1, 2$
Case 5 : $j = 5$ and $R(x, y) = (y, x)$
Case 6 : $j = 5$ and $R(x, y) = (x, -y)$
Case 7 : $j = 5$ and $R(x, y) = (-y, -x)$
Case 8 : $j = 5$ and $R(x, y) = (-x, y)$
Case 9 : $j = 5$ and $r(c) = c * e^{2\pi ik/4}, k = 0, 1, 2, 3$
Case 10 : $j = 6$ and $R(x, y) = (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{\sec^2(\frac{2\pi}{5})})$
Case 11 : $j = 6$ and $r(c) = c * e^{2\pi ik/5}, k = 0, 1, 2, 3, 4$

Lemma 7.3 *For each pair j and R satisfying Lemmas 7.1 and 7.2, we have $z_{n,R(c)} = R(z_{n,c})$*

Proof of Lemma 7.3 : We prove this by induction.

The base case $n = 0$, is clear, since the $0th$ iterate is always 0. Assume it's true for $n = N$ [this is the "inductive step"]. This means, $R(z_{N,c}) = z_{N,R(c)}$.

We have $z_{N+1,R(c)} = z^j_{N,R(c)} + R(c) = R(z_{N,c})^j + R(c)$ (by assumption) $= R(z_{N,c})^j) + R(c) = R(z^j_{N,c} + c) = R(z_{N+1,c})$. Thus the lemma is true by induction. (follows for j and r)

7.2 Rotational Symmetries

In the rotational case, which is easier than the reflectional case, we can actually prove Lemma 7.3 directly. The first step of the rotational symmetry proofs is to assign ω to a $j-1^{th}$ root of unity. We then proceed to iterate ω over the polynomial $f(z) = z^j + c$. After a few iterations, we observe that each iterate rotates the previous one by ω.

Proof of Lemma 7.3 Case 4

$\underline{f(z) = z^4 + c}$

Let ω = $e^{2\pi i/3}$
$f(z) = z^4 + \omega c$

$f(0) = 0^4 + \omega c = \omega c$

$f(\omega c) = (\omega c)^4 + \omega c$
$\omega^3 = 1 \rightarrow \omega^4 = \omega$
$\rightarrow f(\omega c) = \omega c^4 + \omega c$
$= \omega(c^4 + c)$

$f(\omega(c^4 + c)) = (\omega(c^4 + c))^4 + \omega c$
$= \omega^4(c^4 + c)^4 + \omega c$
$= \omega(c^4 + c)^4 + \omega c$
$= \omega[(c^4 + c)^4 + c]$

Proof of Lemma 7.3 Case 9

$\underline{f(z) = z^5 + c}$

Let ω = $e^{2\pi i/4}$
$\rightarrow \omega = i$
$f(z) = z^5 + ic$

$f(0) = 0^5 + ic = ic$

$f(ic) = (ic)^5 + ic$
$i^4 = 1 \rightarrow i^5 = i$ or $\omega^4 = 1 \rightarrow \omega^5 = \omega$
$(ic)^5 = i^5 * c^5 = ic^5$
$\rightarrow f(ic) = ic^5 + ic$
$= i(c^5 + c)$

$f(i(c^5 + c)) = (i(c^5 + c))^5 + ic$
$= i^5(c^5 + c)^5 + ic$

$= i(c^5 + c)^5 + ic$
$= i[(c^5 + c)^5 + c]$

Proof of Lemma 7.3 Case 11

$\underline{f(z) = z^6 + c}$

Let ω = $e^{2\pi i/5}$
$f(z) = z^6 + \omega c$

$f(0) = 0^6 + \omega c = \omega c$

$f(\omega c) = (\omega c)^6 + \omega c$
$\omega^5 = 1 \to \omega^6 = \omega$
$\to f(\omega c) = \omega c^6 + \omega c$
$= \omega(c^6 + c)$

$f(\omega(c^6 + c)) = (\omega(c^6 + c))^6 + \omega c$
$= \omega^6(c^6 + c)^6 + \omega c$
$= \omega(c^6 + c)^6 + \omega c$
$= \omega[(c^6 + c)^6 + c]$

Note : For rotations, Lemma 7.1 is immediately clear by taking absolute values.

Disclaimer : After completing our work, we were informed that Cases 4, 9, and 11 of this lemma are also proved in the blog of Inigo Quilez [12], who is known for their beautiful mathematical visualizations.

7.3 Reflectional Symmetries

The next 16 cases regarding reflectional symmetry (from Lemmas 7.1 and 7.2) are proved by writing the formula for the reflection R in (x, y) form. Using Wolfram Alpha, we checked some of the longer computations in this section.

7.3.1 $f(z) = z^4 + c$

$\underline{y = -\sqrt{3}x}$

$R(x, y) = (\frac{(1-(-\sqrt{3})^2)x+2(-\sqrt{3})y}{1+(-\sqrt{3})^2}, \frac{((-\sqrt{3})^2-1)y+2(-\sqrt{3})x}{1+(-\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 7.1 Case 1

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a - \frac{\sqrt{3}}{2}b)^2 + (-\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 1

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3), -\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3)] + [-\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a - \frac{\sqrt{3}}{2}b, -\frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$
$= ([-\frac{1}{2}a - \frac{\sqrt{3}}{2}b] + [-\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$
$= [-\frac{1}{2}(a^4-6a^2b^2+b^4) - \frac{\sqrt{3}}{2}(4a^3b-4ab^3)] + [-\frac{\sqrt{3}}{2}(a^4-6a^2b^2+b^4) + \frac{1}{2}(4a^3b-4ab^3))]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

$\underline{y = \sqrt{3}x}$

$R(x, y) = (\frac{(1-(\sqrt{3})^2)x+2(\sqrt{3})y}{1+(\sqrt{3})^2}, \frac{((\sqrt{3})^2-1)y+2(\sqrt{3})x}{1+(\sqrt{3})^2})$

$R(x, y) = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$

Proof of Lemma 7.1 Case 2

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$

$|R(z)|$
$= \sqrt{(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b)^2 + (\frac{\sqrt{3}}{2}a + \frac{1}{2}b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 2

$z^4 = (a + bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$
$= (a^4 - 6a^2b^2 + b^4) + (4a^3b - 4ab^3)i$
$= (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3), \frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))$
$= [-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3)] + [\frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))]i$

$(R(z))^4$
$R(z)$
$= R(a, b)$
$= (-\frac{1}{2}a + \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$
$= [-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i$
$(R(z))^4$
$= ([-\frac{1}{2}a + \frac{\sqrt{3}}{2}b] + [\frac{\sqrt{3}}{2}a + \frac{1}{2}b]i)^4$
$= [-\frac{1}{2}(a^4 - 6a^2b^2 + b^4) + \frac{\sqrt{3}}{2}(4a^3b - 4ab^3)] + [\frac{\sqrt{3}}{2}(a^4 - 6a^2b^2 + b^4) + \frac{1}{2}(4a^3b - 4ab^3))]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

$\underline{y = 0}$

$R(x, y) = (\frac{(1-(0)^2)x + 2(0)y}{1+(0)^2}, \frac{((0)^2-1)y + 2(0)x}{1+(0)^2})$

$R(x, y) = (x, -y)$

Proof of Lemma 7.1 Case 3

Let $z = a + bi; (a, b)$
$R(z) = R(a, b) = (a, -b)$

$|R(z)| = |(a, -b)|$
$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 3

$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$

$R(z^4)$
$= R(a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$
$= (a^4 - 6a^2b^2 + b^4, -[4a^3b - 4ab^3])$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$

$(R(z))^4$
$= [a + (-b)i]^4$
$= [a^4 - 6a^2b^2 + b^4] + [-(4a^3b - 4ab^3)]i$
$= R(z^4)$

$R(z^4) = (R(z))^4$

7.3.2 $f(z) = z^5 + c$

$\underline{y = x}$

$R(x, y) = (\frac{(1-(1)^2)x+2(1)y}{1+(1)^2}, \frac{((1)^2-1)y+2(1)x}{1+(1)^2})$

$R(x, y) = (y, x)$

Proof of Lemma 7.1 Case 5

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (b, a)$

$|R(z)|$
$= |(b, a)|$
$= \sqrt{b^2 + a^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 5

$z^5 = (a + bi)^5$
$= a^5 + 5a^4bi - 10a^3b^2 - 10a^2b^3i + 5ab^4 + b^5i$
$= a^5 - 10a^3b^2 + 5ab^4 + 5a^4bi - 10a^2b^3i + b^5i$
$= [a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (5a^4b - 10a^2b^3 + b^5, a^5 - 10a^3b^2 + 5ab^4)$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$

$(R(z))^5$
$= (b + ai)^5$
$= [5a^4b - 10a^2b^3 + b^5] + [a^5 - 10a^3b^2 + 5ab^4]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = 0}$

$R(x, y) = (\frac{(1-(0)^2)x+2(0)y}{1+(0)^2}, \frac{((0)^2-1)y+2(0)x}{1+(0)^2})$

$R(x, y) = (x, -y)$

Proof of Lemma 7.1 Case 6

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (a, -b)$

$|R(z)|$
$= |(a, -b)|$

$= \sqrt{a^2 + (-b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 6

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (a^5 - 10a^3b^2 + 5ab^4, -[5a^4b - 10a^2b^3 + b^5])$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$

$(R(z))^5$
$= [a + (-b)i]^5$
$= [a^5 - 10a^3b^2 + 5ab^4] + (-[5a^4b - 10a^2b^3 + b^5])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{y = -x}$

$R(x, y) = (\frac{(1-(-1)^2)x+2(-1)y}{1+(-1)^2}, \frac{((-1)^2-1)y+2(-1)x}{1+(-1)^2})$

$R(x, y) = (-y, -x)$

Proof of Lemma 7.1 Case 7

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-b, -a)$

$|R(z)|$
$= |(-b, -a)|$
$= \sqrt{(-b)^2 + (-a)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 7

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[5a^4b - 10a^2b^3 + b^5], -[a^5 - 10a^3b^2 + 5ab^4])$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$

$(R(z))^5$
$= [(-b) + (-a)i]^5$
$= -[5a^4b - 10a^2b^3 + b^5] + (-[a^5 - 10a^3b^2 + 5ab^4])i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

$\underline{x = 0}$

$R(x, y)$
$= \lim_{x\to\infty}(\frac{(1-(m)^2)x+2(m)y}{1+(m)^2}, \frac{((m)^2-1)y+2(m)x}{1+(m)^2})$
$= \lim_{x\to\infty}(\frac{(-(m)^2)x+2(m)y}{(m)^2}, \frac{((m)^2)y+2(m)x}{(m)^2})$
$= \lim_{x\to\infty}(-x + \frac{2y}{m}, y + \frac{2x}{m})$
$= (-x, y)$

$R(x, y) = (-x, y)$

Proof of Lemma 7.1 Case 8

Let $z = a + bi; (a, b)$
$R(z)$
$= R(a, b)$
$= (-a, b)$

$|R(z)|$
$= |(-a, b)|$
$= \sqrt{(-a)^2 + (b)^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 8

$z^5 = (a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$

$R(z^5)$
$= R(a^5 - 10a^3b^2 + 5ab^4, 5a^4b - 10a^2b^3 + b^5)$
$= (-[a^5 - 10a^3b^2 + 5ab^4], 5a^4b - 10a^2b^3 + b^5)$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$

$(R(z))^5$
$= (-a + bi)^5$
$= -[a^5 - 10a^3b^2 + 5ab^4] + [5a^4b - 10a^2b^3 + b^5]i$
$= R(z^5)$

$R(z^5) = (R(z))^5$

7.3.3 $f(z) = z^6 + c$

$f(z) = z^6 + c, y = \tan(\frac{2\pi}{5})x$

$R(x, y) = (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{1+\tan^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{1+\tan^2(\frac{2\pi}{5})})$
$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]x+2\tan(\frac{2\pi}{5})y}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]y+2\tan(\frac{2\pi}{5})x}{\sec^2(\frac{2\pi}{5})})$

Proof of Lemma 7.1 Case 10

Let $z = a + bi; (a, b)$
$|R(z)|$
$= |R(a, b)|$
$= |(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})}, \frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})|$
$\sqrt{(\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})})^2 + (\frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})})^2}$
$= \sqrt{a^2 + b^2}$
$= |z|$

$|R(z)| = |z|$

Proof of Lemma 7.2 Case 10

$z^6 = (a + bi)^6$
$= a^6 + 6a^5bi - 15a^4b^2 - 20a^3b^3i + 15a^2b^4 + 6ab^5i - b^6$

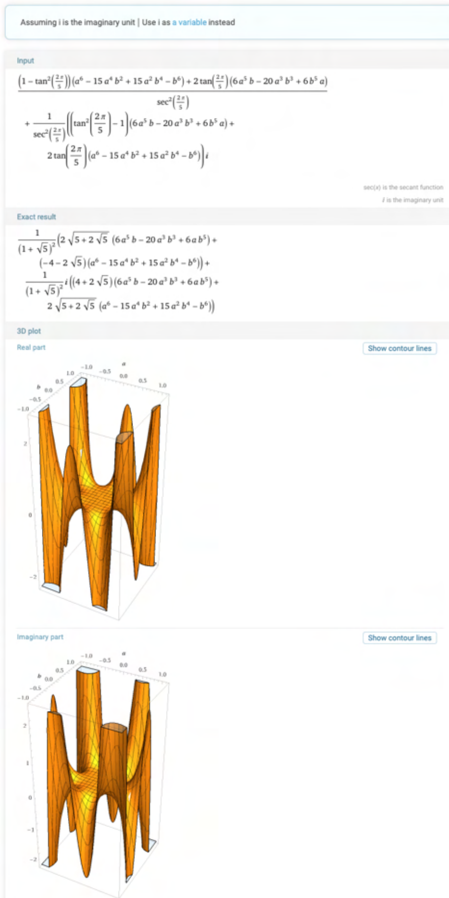
$$= a^6 - 15a^4b^2 + 15a^2b^4 - b^6 + 6a^5bi - 20a^3b^3i + 6ab^5i$$
$$= [a^6 - 15a^4b^2 + 15a^2b^4 - b^6] + [6a^5b - 20a^3b^3 + 6ab^5]i$$
$$= (a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$$

$$R(z^6)$$
$$= R(a^6 - 15a^4b^2 + 15a^2b^4 - b^6, 6a^5b - 20a^3b^3 + 6ab^5)$$
$$= (\frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})},$$
$$\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})})$$
$$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$$
$$+\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$$

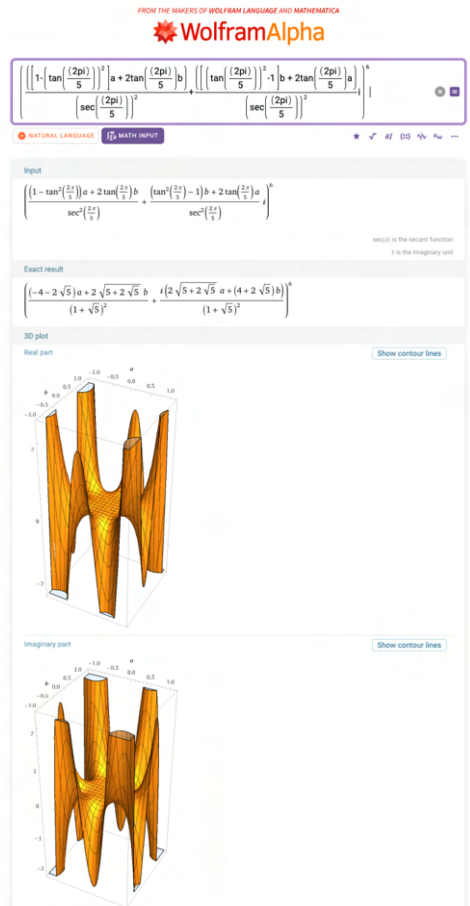
$$(R(z))^6$$
$$= (\frac{[(1-\tan^2(\frac{2\pi}{5})]a+2\tan(\frac{2\pi}{5})b}{\sec^2(\frac{2\pi}{5})} + \frac{[\tan^2(\frac{2\pi}{5})-1]b+2\tan(\frac{2\pi}{5})a}{\sec^2(\frac{2\pi}{5})}i)^6$$
$$= \frac{[(1-\tan^2(\frac{2\pi}{5})](a^6-15a^4b^2+15a^2b^4-b^6)+2\tan(\frac{2\pi}{5})(6a^5b-20a^3b^3+6ab^5)}{\sec^2(\frac{2\pi}{5})}$$
$$+\frac{[\tan^2(\frac{2\pi}{5})-1](6a^5b-20a^3b^3+6ab^5)+2\tan(\frac{2\pi}{5})(a^6-15a^4b^2+15a^2b^4-b^6)}{\sec^2(\frac{2\pi}{5})}i$$
$$= R(z^6)$$

$$R(z^6) = (R(z))^6$$

While equality may not seem immediately obvious due to its large computational nature, the corresponding contour plots of $R(z)^6$ and $(R(z))^6$ match, strongly suggesting equality.



$R(z)^6$



$(R(z))^6$

Theorem 7.4 *For each pair j and R satisfying Lemmas 1 and 2, if c is in the jth generalized Mandelbrot set, then $R(c)$ is in the jth generalized Mandelbrot set.*

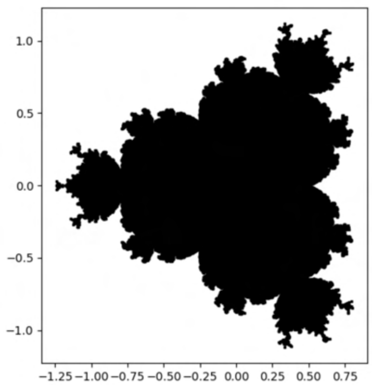
Proof of Theorem 7.4

Since c is in the jth-generalized Mandelbrot set, by definition the iterates $z_{n,c}$ satisfy $|z_{n,c}| < B$ for some B and all n

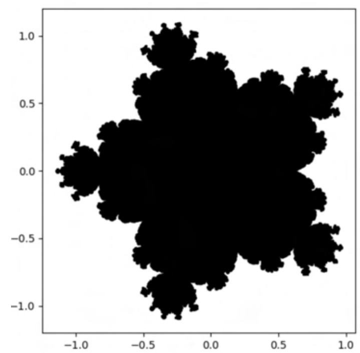
By Lemma 7.1 and 7.2, we have $|z_{n,R(c)}| = |R(z_{n,c})| = |z_{n,c}| < B$. Thus $R(c)$ is in the jth Mandelbrot set by definition. (follows for j and r)

8 Unique Properties

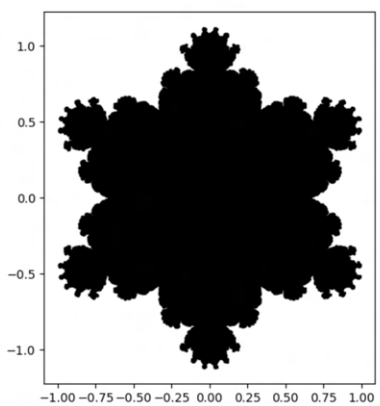
Observe that for higher degree polynomials, there appears to be a number of regions, or "petals" formed. Specifically, for the polynomial $f(z) = z^j + c$, there are $j - 1$ petals.



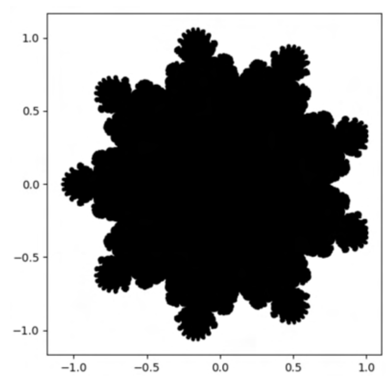
$f(z) = z^4 + c$
3 petals



$f(z) = z^6 + c$
5 petals



$f(z) = z^7 + c$
6 petals



$f(z) = z^{10} + c$
9 petals

The petals clearly reflect the rotational and reflectional symmetries we proved earlier in Section 7, but have other striking properties, which would be nice to formalize in the future.

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j	Area
3	1.8966
4	2.0533
5	2.1717
6	2.2628
7	2.3357
8	2.3966
9	2.4470
10	2.4897
11	2.5269
12	2.5595
13	2.5884
14	2.6137
15	2.6370
16	2.6578
17	2.6764
18	2.6939
19	2.7104
20	2.7249

Table 2: Pixel density = 512, iterations = 25

j	Area
3	1.8161
4	1.9971
5	2.1271
6	2.2628
7	2.3054
8	2.3699
9	2.4239
10	2.4687
11	2.5077
12	2.5419
13	2.5723
14	2.5988
15	2.6227
16	2.6451
17	2.6647
18	2.6829
19	2.6994
20	2.7145

Table 3: Pixel density = 512, iterations = 100

j	Area
3	1.7969
4	1.9834
5	2.1162
6	2.2182
7	2.2976
8	2.3629
9	2.4173
10	2.4632
11	2.5025
12	2.5368
13	2.5676
14	2.5944
15	2.6186
16	2.6413
17	2.6612
18	2.6796
19	2.6962
20	2.7117

Table 4: Pixel density = 512, iterations = 500

j	Area
3	1.7954
4	1.9819
5	2.1149
6	2.2169
7	2.2967
8	2.3621
9	2.4165
10	2.4624
11	2.5017
12	2.5361
13	2.5671
14	2.5939
15	2.6181
16	2.6407
17	2.6608
18	2.6790
19	2.6958
20	2.7113

Table 5: Pixel density = 512, iterations = 1000