

Knots Arising From the Planar Equal-Mass Collisionless Three-Body Problem

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Abstract

We present a series of steps that transform equal-mass collisionless planar three-body configurations into closed curves in \mathbb{R}^3 . Exploiting a centre-of-mass constraint and a size constraint, we reduce the original configuration in \mathbb{R}^6 to \mathbb{R}^4 . We then use a rotation and an additional normalisation to place these configurations on the 3-sphere S^3 , followed by a stereographic projection onto \mathbb{R}^3 . This procedure turns a periodic orbit into a closed curve that can be classified as a knot. We apply our method to the fifteen periodic configurations presented by Šuvakov and Dmitrašinović and show that most of these trajectories become the unknot under our projection. However, three orbits become trefoils and one becomes a figure-eight. We discuss possible directions for future work.

Introduction

This paper answers two questions:

1. Can we find an orbit in each of the classes listed in the paper of Šuvakov and Dmitrašinović which is non-trivially knotted? No.
2. Can two orbits with the same free group element have different knot types? Yes.

The classical three-body problem has been studied by mathematicians and physicists since Newton found the closed-form for the two-body problem. In the 19th century, Poincaré showed that the general three-body problem is non-integrable [3]. Nonetheless, families of periodic orbits have been discovered and studied for their insights into n -body systems. The planar case specifically is interesting to study as the planets of the solar system have been shown to lie in a plane [4][6].

In 2013, Šuvakov and Dmitrašinovic [5] made a significant advance by finding thirteen distinct collisionless periodic orbits of three unit masses confined to a plane, out of fifteen total examples (two already discovered). Since then, hundreds more planar periodic orbits have been discovered [1].

Montgomery, in [2], mapped planar configurations to the shape sphere $S^2 \subset \mathbb{R}^3$. One step involved contracting $S^3 \subset \mathbb{R}^4$ to S^2 via a Hopf fibration. However, some information about the orbit is lost this way. A promising approach to studying these orbits is to rather represent $S^3 - \{0, 0, 0, 1\}$ as \mathbb{R}^3 , therefore preserving information. Motivated by Montgomery's work, we propose a four-step procedure. Concretely, we start with the space of all reduced planar configurations, then:

1. Impose a centre-of-mass constraint, reducing dimension by 2.

2. Perform a linear transformation to diagonalise the quadratic form which emerges from the previous mapping.
3. Normalise the surface to S^3 .
4. Stereographically project S^3 onto \mathbb{R}^3 .

By following a periodic solution in time, the image of these maps becomes a closed curve in \mathbb{R}^3 whose knot type can be identified. In this paper, we describe our mappings and illustrate their application to the fifteen orbits of Šuvakov and Dmitrašinović [5]. We then provide a table summarising the orbit classification results. We conclude by highlighting possible future work.

Assumptions and Constraints

We begin with three points masses of unit mass, indexed by $i = 1, 2, 3$ each associated with a position vector $\vec{P} = (x_i, y_i)$ in \mathbb{R}^2 . Thus, the initial configuration space is \mathbb{R}^6 . We consider the set of configurations which adhere to the following two constraints.

1. Centre-of-mass constraint:

$$\sum_{i=1}^3 \vec{P}_i = (0, 0).$$

2. Size constraint:

$$\sum_{i=1}^3 \|\vec{P}_i\|^2 = 1.$$

Note that any three vectors may be scaled down such that they satisfy the size constraint by redefining the points as the following.

$$\vec{P}'_i = \frac{\vec{P}_i}{\sqrt{\sum_{i=1}^3 \|\vec{P}_i\|^2}}.$$

The set of all such $(\vec{P}'_1, \vec{P}'_2, \vec{P}'_3)$ adhering to both constraints lies on a 4 dimensional surface.

The Procedure

The four mappings are summarised in the following diagram.

$$\mathbb{R}^6 \xrightarrow{I} \mathbb{R}^4 \xrightarrow{II} \mathbb{R}^4 \xrightarrow{III} S^3 \xrightarrow{IV} \mathbb{R}^3$$

Step I: Fixing Centre of Mass

Since

$$\sum_{i=1}^3 \vec{P}_i = (0, 0).$$

we rearrange for \vec{P}_3 :

$$\vec{P}_3 = -(\vec{P}_1 + \vec{P}_2).$$

As a result, every configuration is determined by \vec{P}_1 and \vec{P}_2 alone. By the size constraint,

$$\|\vec{P}_1\|^2 + \|\vec{P}_2\|^2 + \|\vec{P}_3\|^2 = \|\vec{P}_1\|^2 + \|\vec{P}_2\|^2 + \|- \vec{P}_1 - \vec{P}_2\|^2 = 1.$$

Equivalently,

$$2x_1^2 + 2x_1x_2 + 2x_2^2 + 2y_1^2 + 2y_1y_2 + 2y_2^2 = 1.$$

Hence, Mapping I sends $(\vec{P}_1, \vec{P}_2, \vec{P}_3) \in \mathbb{R}^6$ to $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$, subject to the above quadratic constraint.

Step II: Diagonalising the Quadratic Form

Set $\vec{X} = (x_1, x_2)$ and $\vec{Y} = (y_1, y_2)$. The previous constraint shows that

$$\vec{X}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{X} + \vec{Y}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{Y} = 1.$$

Call the shared 2×2 matrix A . We see that A has eigenvalues $\lambda_1 = 3$ and $\lambda_3 = 1$. To diagonalise, we must find a θ such that the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

satisfies

$$R(\theta)^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} R(\theta) = \text{diag}(3, 1).$$

Recall that θ must satisfy

$$\tan(2\theta) = \frac{a_{12} + a_{21}}{a_{11} - a_{22}} = \frac{2}{0}.$$

Thus $\theta = \pi/4$, which in turn means

$$R = R\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore, the new basis vectors are

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = R^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = R^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which satisfy

$$(u_1 \ u_2)A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3u_1^2 + u_2^2$$

and

$$(v_1 \ v_2)A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3v_1^2 + v_2^2.$$

The constraint becomes

$$3u_1^2 + u_2^2 + 3v_1^2 + v_2^2 = 1.$$

Therefore, Step II places the configuration on a hyperellipsoid in \mathbb{R}^4 .

Step III: Normalising onto S^3

We perform a change of variables to map the quadruple (u_1, u_2, v_1, v_2) onto $S^3 \subset \mathbb{R}^4$. Define

$$(s_1, s_2, t_1, t_2) = (\sqrt{3}u_1, u_2, \sqrt{3}v_1, v_2).$$

By construction, we have that

$$s_1^2 + s_2^2 + t_1^2 + t_2^2 = 1.$$

Hence, (s_1, s_2, t_1, t_2) must lie on S^3 .

Step IV: Stereographic Projection of $S^3 - \{0, 0, 0, 1\}$ to \mathbb{R}^3

The trajectory crosses the point $(0, 0, 0, 1)$ only when there is a collision between two of the bodies. Thus, we may remove the point $(0, 0, 0, 1)$ and project the remainder to \mathbb{R}^3 . Formally, let

$$(s_1, s_2, t_1, t_2) \in S^3 - \{(0, 0, 0, 1)\}.$$

We define

$$(x, y, z) = \left(\frac{s_1}{1-t_2}, \frac{t_1}{1-t_2}, \frac{s_2}{1-t_2} \right).$$

This yields a bijection from $S^3 - \{(0, 0, 0, 1)\}$ to \mathbb{R}^3 . Therefore, under the compositions of Steps I-IV, each configuration $(\vec{P}_1, \vec{P}_2, \vec{P}_3)$ becomes a closed loop in \mathbb{R}^3 .

Closed Loops and Knot Classification

Let $L = (\vec{P}_1(t), \vec{P}_2(t), \vec{P}_3(t))$ be a periodic collisionless planar orbit with period $T > 0$ which adheres to the centre-of-mass condition and the size condition. Represent L as (x_1, y_1, x_2, y_2) , as described in Step I. Under the mappings presented in Step II, Step III and Step IV, we are given a continuous curve

$$C: [0, T] \rightarrow \mathbb{R}^3,$$

where

$$C(t) = (IV \circ III \circ II)(x_1, y_1, x_2, y_2),$$

mappings *II*, *III*, and *IV* represent the mappings outlined in Step II, Step III, and Step IV respectively, and $C(0) = C(T)$. We can now classify C as a knot.

Application to Šuvakov and Dmitrašinovic's Orbits.

We attempt to classify the trajectories which Šuvakov and Dmitrašinović discovered in [5]. We first numerically approximated each configuration with a Runge-Kutta algorithm with linear step size 0.0001 until a period elapsed. Restriction to one period made error and deviation practically negligible. Subsequently, we translated and scaled the collection of positions to follow the size and centre-of-mass constraints before applying the mappings in Step II, Step III and Step IV. To

determine knot type, we projected the orbit to an orthogonal unit basis, then applied Reidemeister moves.

Some orbits are easily determinable, such as I.A.2 butterfly II, while others are more difficult, such as I.A.1 butterfly I. Both trajectories are shown in Figure 1. Figure 2 shows how a curve can be simplified using Reidemeister moves.

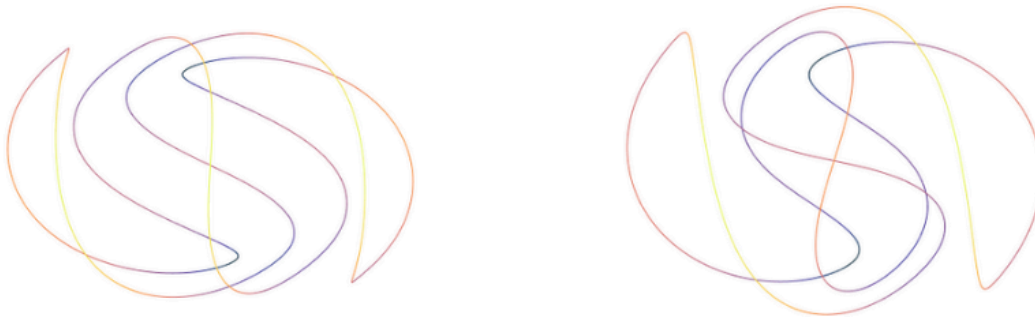


Figure 1. I.A.2 butterfly II (left) and I.A.1 butterfly I (right). Lighter colours denote higher elevation (when viewed in 3D after Step IV).

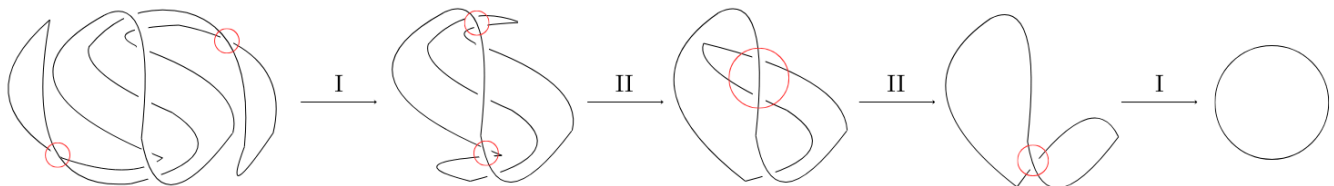


Figure 2. I.A.2 butterfly II being deformed into the unknot. The red circles show the crossings being acted upon. Operations I, II, and III represent their corresponding Reidemeister moves.

The following table summarises our classifications of each orbit. We include the initial velocity parameters, $\dot{x}(0)$, $\dot{y}(0)$, the free group element, as given in [5], and the resulting knot type. We see that class II.C does not have an orbit which is non-trivial—thus, not every class has a non-trivial knot. We also notice that I.A.1 butterfly I and I.A.2 butterfly II have the same free-group element, yet have different knot types, therefore answering both of our questions.

Class, Number, and Name	$\dot{x}(0)$	$\dot{y}(0)$	Free Group Element	Knot Classification
I.A.1 butterfly I	0.30689	0.12551	$(ab)^2(AB)^2$	trefoil
I.A.2 butterfly II	0.39295	0.09758	$(ab)^2(AB)^2$	unknot

I.A.3 bumblebee	0.18428	0.58719	1	unknot
I.B.1 moth I	0.46444	0.39606	$ba(BAB)ab(ABA)$	figure-eight
I.B.2 moth II	0.43917	0.45297	$(abAB)^2 A (baBA)^2$	unknot
I.B.3 butterfly III	0.40592	0.23016	$(ab)^2(ABA)(ba)^2(BAB)$	unknot
I.B.4 moth III	0.38344	0.37736	$(babABA)^2 a(abaBAB)^2 b$	unknot
I.B.5 goggles	0.08330	0.12789	$(ab)^2 ABBA(ba)^2 BAAB$	trefoil
I.B.6 butterfly IV	0.350112	0.07934	$((ab)^2(AB)^2)^6 A((ba)^2(BA)^2)^6 B$	unknot
I.B.7 dragonfly	0.08058	0.58884	$(b^2(ABabAB))(a^2(BAbaBA))$	unknot
II.B.1 yarn	0.55906	0.34919	$(babABabaBA)^3$	trefoil
II.C.2a yin-yang I	0.51394	0.30474	$(ab)^2(ABA)ba(BAB)$	unknot
II.C.2b yin-yang I	0.28270	0.32721	$(ab)^2(ABA)ba(BAB)$	unknot
II.C.3a yin-yang II	0.41682	0.33033	$(abaBAB)^3(abaBAbab)(ABAbab)^3(AB)^2$	unknot
II.C.3b yin-yang	0.41734	0.31310	$(abaBAB)^3(abaBAbab)(ABAbab)^3(AB)^2$	unknot

Table 1. Initial conditions of the three-body orbits and knot classification. Labelling for the orbits follow what is written in [5]. Quantities $\dot{x}_i(0)$ and $\dot{y}_i(0)$ are the i th body's initial velocities in the x and y direction, respectively. The values $x_i(0)$ and $y_i(0)$ are the i th body's positions in the x direction and y direction, respectively. The initial velocities of the bodies are $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}(0)$, $\dot{x}_3(0) = -2\dot{x}(0)$, $\dot{y}_1(0) = \dot{y}_2(0) = \dot{y}(0)$, $\dot{y}_3(0) = -2\dot{y}(0)$. The initial positions of the bodies are $x_1(0) = -x_2(0) = -1$, $x_3(0) = 0$, $y_1(0) = y_2(0) = y_3(0) = 0$. The gravitational constant G is taken to be 1 and the masses m_1, m_2, m_3 are equal and taken to be 1.

$${}^1(b(ABab)^2 A^2 (baBA)^2 ba) (B^2 (abAB)^2 a^2 (BAba)^2 BA)$$

Conclusion

We presented a series of four consecutive mappings which turn a periodic three-body planar equal-mass system into a closed loop in \mathbb{R}^3 , allowing the orbit to be classified as a knot.

Afterwards, we applied the mappings to the fifteen trajectories Šuvakov and Dmitrašinović displayed in [5], showing that the majority of the orbits can be classified as the unknot. But, three yield the trefoil knot and another produces the figure-eight, showing that nontrivial knottedness is possible.

Future work could be dedicated to applying the mappings described previously to more periodic orbits, such as the results presented in [1], to see if more complicated knot types emerge or proving that only a certain set of knots are able to appear. Another avenue to consider would be to generalise the mappings to non-identical masses.

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