Effect of Physical Mechanics on Development and Construction of a Space Elevator in The Modern Era

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Abstract

Born in science fiction but becoming increasingly real, the space elevator, if possible, is a compelling alternative to the inefficiencies of modern staged rockets. A space elevator is simply a physical connection between a point on the Earth and a counterweight in geostationary orbit, along which cargo can travel. This has several goals: to increase our capacity to access the resources of our solar system, to reduce the environmental strain we exert on our planet, and to further the various component fields of science necessarily involved in such an effort. A space elevator is at once very simple and very complex. Although the underlying principles of a taut cable with objects able to move along it are relatively intuitive, the stress placed on the cable is immense, and designing a material capable of withstanding the pressure without becoming prohibitively bulky through tapering is a challenge that we will assess. The purpose of this paper is to examine current methodology of development, illustrate the forces that would act on the space elevator system, and evaluate the feasibility and timeline of the device. First, a background of basic celestial mechanics will be provided, following Keplerian mechanics, with the two body assumption that only the gravitational effects of the Earth must be accounted for. With that, a simplified gravity gradient model will be introduced to identify the moments on the spacecraft and inform the design of the counterweight system. Finally, this paper will identify the material and structural design necessary to construct a space elevator and its manufacturing feasibility.

Station Background

The idea of a space elevator is a tethered space station anchored on Earth and extending out past the atmosphere in order to facilitate transport between Earth and space. The possibilities of what could be done with a space station that has access to the full scope of Earth's resources broaden immensely. Compared to a rocket's ascent, any mechanical elevator would have a relatively slow and smooth journey. The cost of a journey per unit of mass would also steeply drop, as the fuel required would be negligible in comparison to the fuel for typical launch providers. In FY2000 dollars, the estimated total cost to deliver a satellite to Geostationary Orbit was roughly $214M onboard a Titan IV launch vehicle [1]. For scientific experimentation, this is critical, because the two barriers to space experimentation are the prohibitive cost of, for instance, setting up a lab to study long term effects of outer space life on mice, and that the journey itself would be impossible with volatile or delicate materials. Additionally, access to orbit on the industrial scale would completely transform the world of spaceflight. Modern spacecraft's carrying capacity and size are largely constrained to combat the challenge of getting out of the atmosphere. Beginning from orbit would cost-effectively bring into reach our solar system and beyond for spectacular and specialized new forms of spacecraft.

Orbital Mechanics

In order for space elevators to function properly, the elevator's station must stay directly above its relevant ground station on Earth. If the satellite counter weight is placed too far or too close to the Earth, this will cause the counterweight, and therefore the tension cable itself, to drift and eventually wrap around the Earth, a dire risk that must be avoided. In other words, its orbital period must match the time it takes for Earth to complete a 360º rotation, or 1 day. Assuming classical orbital mechanics in the Keplerian domain, the period of a spacecraft's orbit may be derived for Eq. 1 below. Where the *mu* term is the gravitational constant of the Earth and the *r* term is the radius from the center of the Earth, to the spacecraft in a circular orbit. The circular orbit assumption is valid here since the authors are only considering placement of the space elevator in an orbit where the angular rates would be nearly constant, a requirement of the system.

$$T_{Circ} = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$

$$\mu_E = 3.986 \times 10^{14} \frac{m^3}{s^2}$$

Defining T as 86,164 seconds and assuming a circular orbit, the correct distance from the center of the earth derived from the equation above is found to be geostationary orbit (GEO) at 42,164 km from the core. This is where the station and counterweight will be located.

Force Analysis

In a space elevator model, the cable is placed under extreme stresses due to the immense forces of tension acting along its length. Neglecting air friction and a safety margin the stress on the cable at geostationary orbit may be derived by first setting equal the forces present on the system [2]. Those forces being the stress differential across the cross sectional area, force of gravity, and the centripetal acceleration force for a differential distance from the central body.

$$AdT = \frac{GM(Adr\rho)}{r^2} - (Adr\rho)\omega^2 r$$

Dividing both sides by Adr provides

$$\frac{dT}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_g^{\ 3}}\right)$$

Integrating from the surface of the Earth (R) to GEO radius Rg,

$$\int dT = \int_R^{R_g} GM\rho\left(\frac{1}{r^2} - \frac{r}{R_g^{\ 3}}\right)dr$$

$$T(R_g) = GM\rho\left[\int_R^{R_g}\frac{1}{r^2}\,dr - \int_R^{R_g}\frac{r}{R_g^{\ 3}}\,dr\right]$$

$$T(R_g) = GM\rho\left[-\frac{1}{r}\Big|_R^{R_g} - \frac{1}{2}\left(\frac{r^2}{R_g^{\ 3}}\right)\Big|_R^{R_g}\right]$$

This provides the final equation for the tensile strength at GEO as

$$T\left(R_g\right) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_g} + \frac{R^2}{2R_g^3}\right]$$

As an example of the tension present, given a material density value for ρ, the radius in the equation above may be varied to realize the tension at that point. An example case for the carbon nanotube density of $1740\ \frac{kg}{m^3}$ may be found in the figure below by integrating from the surface of Earth to GEO radius.

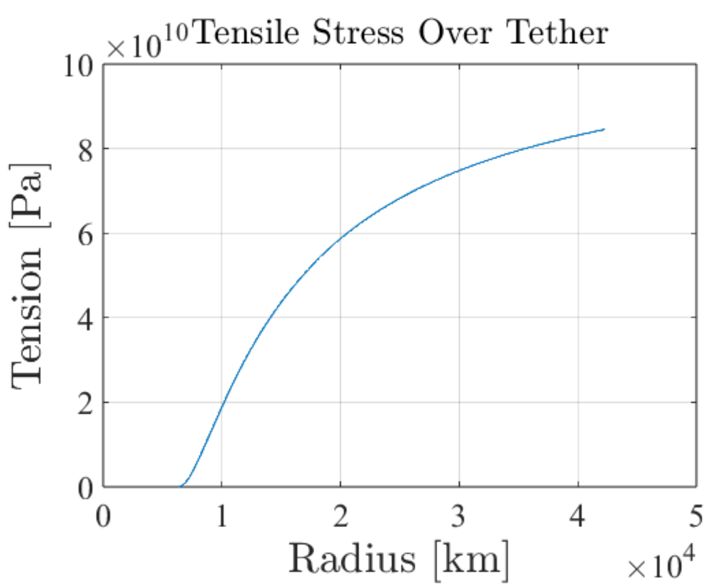

Figure 1: Stress Magnitude Over Carbon Nanotube Tether Radius

As may be seen in Fig. 1, the tensile strength required for a single tether presents enormous stress on the system beyond the capabilities of any material available. In addition, it may be noted in the figure above that the stress remains at a constant value of zero until the radius reaches the radius of the Earth, since the tether must attach to a ground station on the surface of the planet.

Cable Tapering

No material in production or currently under development has the high tensile strength and low density to allow for a cable of uniform circumference stretching between the suggested location of the counterweight and the Earth's surface. Tapering the cable allows for reinforcement in high-stress areas without making the cable as a whole prohibitively bulky. Should tapering be required, the difference between the bottom and top areas of an infinitesimally small dr is described by

$$\frac{dA}{A} = \frac{\rho g R^2}{T}\left[\frac{1}{r^2} - \frac{r}{R_g^{\,3}}\right]dr$$

Integrate from the surface of the planet to GEO provides the required area at any radius.

$$\int_{A_s}^{A(r)} \frac{dA}{A} = \int_{R}^{r} \frac{\rho g R^2}{T}\left[\frac{1}{r^2} - \frac{r}{R_g^{\,3}}\right]dr$$

$$ln(A(r)) - ln(A_s) = \frac{\rho g R^2}{T}\left[-\frac{1}{r}\Big|_R^r - \frac{1}{2}\left(\frac{r^2}{R_g^{\,3}}\right)\Big|_R^r\right]$$

$$ln\left(\frac{A(r)}{A_s}\right) = \frac{\rho g R^2}{T}\left[-\frac{1}{r} + \frac{1}{R} - \frac{1}{2}\left(\frac{r^2}{R_g^{\,3}} - \frac{R^2}{R_g^{\,3}}\right)\right]$$

Canceling the natural log via exponentiation provides

$$\frac{A(r)}{A_s} = exp\left[\frac{\rho g R^2}{T}\left[-\frac{1}{r} + \frac{1}{R} - \frac{1}{2}\left(\frac{r^2}{R_g^{\,3}} - \frac{R^2}{R_g^{\,3}}\right)\right]\right]$$

Simplification provides the final equation below which describes the area of a cross section of the cable A given a density at a point along the cable r necessary to endure the tension at that radius.

$$A(r) = A_s exp\left[\frac{\rho g R^2}{T}\left[\frac{1}{R} + \frac{R^2}{2R_g^3} - \frac{1}{r} - \frac{r^2}{2R_g^3}\right]\right]$$

It may be noted that the above derivations do not account for specific material qualities outside of density. Therefore, it may be more accurate to define the taper ratio according to Misra and Cohen [3]. This definition provides additional insight because it also includes the stretching that will occur under stress. Introducing the characteristic height $\bar{h} = \frac{\sigma_0}{\gamma g_0}$ and $\epsilon_0 = \frac{\sigma_0}{E}$, the taper ratio may be expressed by the equation below.

$$TR = exp\left[\frac{R}{(\bar{h} + \epsilon_0)}\left(1 - \frac{R}{R_G}\right)^2\left(1 + \frac{R}{2R_G}\right)\right]$$

Counterweight Addition

While the above derivations account for the mass of the cable used, a more efficient means of balancing the experienced forces is to employ a counterweight [3]. By attaching a mass to the end of the cable that extends beyond GEO, the overall length of the cable may be reduced and the total required mass of the system may also be reduced. Integrating the previous equations results in the boundary condition that the cross sectional area is zero at the end of the cable. However, it is clear that the area of the cross section of the ribbon cannot be zero at any location for this case of constant stress. Thus, to satisfy the boundary condition at the tip of the ribbon, a mass *mc* (the counterweight) must be attached there. The forces acting on the counterweight can be made equal to the tension at the tip by forcing,

$$m_c(\Omega^2(R_E + L) - \mu/(R_E + L)^2) = \sigma_0 A(s)|_{s = L_0}$$

Where Ω is the rotational velocity found earlier and the right side of the equation is equivalent to the tension in the cable at that point. Augmenting the previous equation to account for the taper ratio and the nominal strain present in the system, denoted $(1 + \epsilon_0)$, the adjusted mass of the counter weight is modeled by

$$m_c = \frac{\sigma_0 A_m exp(F(s))|_{s=L_0}}{\Omega^2(R_E + L_0(1+\epsilon)) - \frac{\mu}{(R_E + L_0(1+\epsilon))^2}}$$

Modeling the counterweight mass as above offers the advantage of showing how the different masses differ as the length of the cable changes. In addition, the total mass can be represented in the figure below. In the figure below, a ribbon with $L_0 = 100,000\ km$ and $A_m = 10mm^2$ would have a mass of about 1,000 tons, assuming a taper ratio of 6x. The corresponding counterweight mass would be about 300 tons. This is beneficial since mass put into orbit offers one of the largest constraints to installing a system like the space elevator.

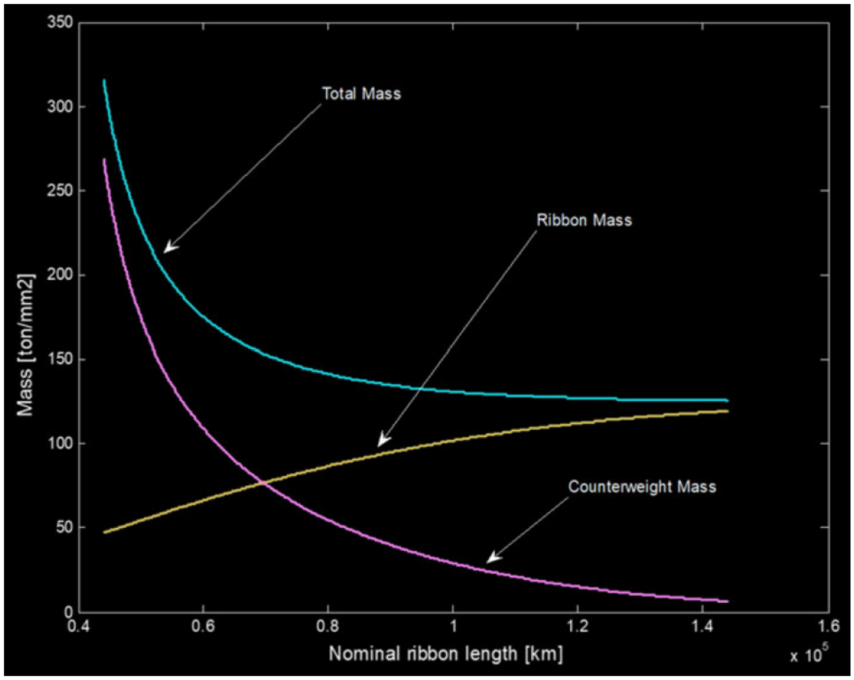

Figure 2: Mass/Maximum Cross Sectional Area vs Tether Length [3]

Available Materials

By generous usage of tapering, construction of a space elevator could begin today, with current materials. However, any mass-producible material in existence today would require an unrealistic amount of tapering to maintain structural integrity. Carbon nanotubes, which could bear the load with a realistic tapering factor, are not currently produced in sufficient quantities to build a space elevator.

Name of Material	Carbon Nanotubes	Steel	Concrete	Rubber
Density	1.74 g/cm^3	7.9 g/cm^3	2.4 g/cm^3	1.34 g/cm^3
Tensile Stress at GEO	$8.43 * 10^{13}$	$3.83 * 10^{14}$	$1.16 * 10^{14}$	$6.50 * 10^{13}$
Stress at Earth	35.2 GPA	726 GPA	66 GPA	20.8 GPA
Yield Strength	$150\ GPA$	$5\ GPA$	$5\ MPA$	$40\ MPA$

Young's Modulus of Elasticity (E)	$1000\ GPA$	$200\ GPA$	$40\ GPA$	$104\ N/m^2$
Taper Factor	6.78	$5.7 * 10^{24}$	-	-

The null spaces above are due to values being too large for standard 32 bit calculation. As can be seen in the table above, the only material tested which could satisfy the constraints of stress remaining below the maximum tensile strength is the carbon nanotube.

Timeline

The exactitudes for the roadmap of development for the space elevator are hotly debated. One corporation, OBAYASHI, estimates that they will complete the entire undertaking by 2050. Others argue that due to a lack of mass-production methods for carbon nanotubes or other suitable materials, a space elevator will simply never be practical. In order to reduce the bottleneck effects of limits on production of carbon nanotubes, a counterweight could be employed as discussed earlier. This, in combination with reasonable growth in carbon nanotube production, could make the space elevator possible. However, securing a celestial object or launching sufficient mass for a counterweight would present challenges of its own. Overall, a space elevator is not likely to be constructed by the earliest estimates of 2050, but has a significant likelihood of becoming technologically possible in this century considering the high likelihood that space will continue to be an area of technological innovation.

Conclusion

This paper provides an overview of space elevators, a groundbreaking technology with the potential to transform satellite placement in orbit and propel humanity into an unprecedented position of access to the solar system and beyond. It explores the achievable orbital parameters through space elevators and offers a concise analysis of various factors, including ribbon length, material stress, cross-sectional area variation, and counterweight mass. These factors are crucial for designing an effective space elevator. In comparison to previous works on the same subject, this paper incorporates previously neglected elements such as calculations for the mass of the counterweight. The future of space elevators is compelling, despite its low workability today. More work to explore the potential of carbon nanotubes and design of a space elevator's climbers could play an important role in the future of both the field and the species.

References

[1]: Larson, W J, and Wertz, J R. Space Mission Analysis and Design Third Ed. United States: N. p., 1999. Text.

[2]: Aravind, P. K. "The physics of the space elevator." American Journal of Physics 75.2 (2007): 125-130.

[3]: Dixit, Uday S., Santosha Kumar Dwivedy, and T. W. Forward. Mechanical Sciences. Springer: Singapore, 2020.