Exploring the Links between Quantum Entanglement and Curved Space Time

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Abstract

This review article discusses the technical derivation and components of the ER=EPR conjecture [arXiv:1306.0533] by Susskind and Maldacena, starting with the mathematical derivation of Einstein-Rosen wormhole solution in general relativity, building on the Schwarzschild metric to arrive at the Eddington-Finklestein and Kruskal-Szekeres coordinates, constructing a Penrose diagram demonstrating the quadripartite model of spacetime. Finally, we introduce elementary ideas from quantum entanglement (EPR) and connect them to the Ryu Takayanagi Formulate [arXiv:hep-th/0603001], and to the Susskind-Maldacena hypothesis of ER=EPR [arXiv:1412.8483].

Contents

1 General Relativity Foundations

1.1 A Conceptual Explanation of GR in ER=EPR

As we are all familiar, Einstein's theory of general relativity was undoubtedly one of the most paradigm shifting discoveries of the 20th century, providing us with the most useful (albeit rigorous) mathematical tools for predicting and calculating the behavior of a curved space time on a cosmic scale.

The Einstein-Rosen Bridge is most accurately described as a structure connecting two maximally distal portions of space, and has been heavily speculated about by science-fiction lovers and theoretical physicists alike. From the most basic conceptual standpoint.

From this point forward, we will work our way through the mathematical foundations for the concepts referenced above, beginning with a derivation of the Einstein Field Equations and working our way through potential solutions to arrive at a proof of the mathematical existence of ERBs. We will then draw the parallels and connections required to develop a truly in-depth understanding of ER=EPR, before considering its numerous implications.

This process presupposes a foundational knowledge of linear algebra, single variable and multi-variable calculus, and basic tensor calculus. Any mathematics referenced beyond the scope of these subjects will be explained in thorough detail.

1.2 Prerequisite Mathematics

This section of the article will provide a terse overview of the basic mathematical tools used in the work. For more details, consult any number of online sources or take the prerequisite course associated with the subject matter. The table below should provide the reader with an adequate refresher to familiar mathematical terminology:

Verbiage	Symbol	Formula
Covariant Derivative	$\nabla_{c^i}(\vec{\mathbf{v}})$	$(\frac{\partial v^k}{\partial c^i} + \Gamma^k_{ij})\vec{\mathbf{e}}_k$
Metric Tensor	g	$\begin{bmatrix} \vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_i & \vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j \\ \vec{\mathbf{e}}_j \cdot \vec{\mathbf{e}}_i & \vec{\mathbf{e}}_j \cdot \vec{\mathbf{e}}_j \end{bmatrix}$
Geodesic Equation	$\frac{\mathrm{d}^2 R}{\mathrm{d}\lambda^2}$	$\frac{\partial^2 u^k}{\partial \lambda^2} + \Gamma^k_{ij} \frac{\mathrm{d}u^i}{\mathrm{d}\lambda} \frac{\mathrm{d}u^j}{\mathrm{d}\lambda}$
Parallel Transport	N/A	$\nabla_{\vec{\mathbf{w}}}(\vec{\mathbf{v}}) = 0$
Riemann Curvature Tensor	$R(\vec{\mathbf{u}}, \vec{\mathbf{v}})\vec{\mathbf{w}}$	$\nabla_{\vec{\mathbf{u}}}\nabla_{\vec{\mathbf{v}}}\vec{\mathbf{w}} - \nabla_{\vec{\mathbf{v}}}\nabla_{\vec{\mathbf{u}}}\vec{\mathbf{w}} - \nabla_{[\vec{\mathbf{u}}.\vec{\mathbf{v}}]}\vec{\mathbf{w}}$
Basis Vectors	$\vec{\mathbf{e}}_i$	$\frac{\partial}{\partial \vec{\mathbf{e}}_i s}$

1.2.1 The Connection Coefficients

With these fundamentals, we may proceed to our derivation of the Connection Coefficients, a mathematical array of numbers crucial to the understanding of the metric tensor.

1. We begin with the assumption of a torsion-free connection.

$$\nabla_{\vec{\mathbf{e}}_i}(\vec{\mathbf{e}}_j) - \nabla_{\vec{\mathbf{e}}_j}(\vec{\mathbf{e}}_i) = [\vec{\mathbf{e}}_i, \vec{\mathbf{e}}_j]$$

   As our basis vectors are partial derivative operators and the order of differentiation does not matter:

$$[\vec{\mathbf{e}}_i, \vec{\mathbf{e}}_j] = \frac{\partial}{\partial \vec{\mathbf{e}}_j}(\vec{\mathbf{e}}_i) - \frac{\partial}{\partial \vec{\mathbf{e}}_i}(\vec{\mathbf{e}}_j) = 0$$
$$\Rightarrow \nabla_{\vec{\mathbf{e}}_i}(\vec{\mathbf{e}}_j) = \nabla \vec{\mathbf{e}}_j(\vec{\mathbf{e}}_i).$$

   We may then expand out these terms in their Christoffel symbol formulas:

$$\nabla_{\vec{\mathbf{e}}_i}(\vec{\mathbf{e}}_j) = \Gamma^k_{ij}\vec{\mathbf{e}}_k$$
$$\nabla_{\vec{\mathbf{e}}_j}(\vec{\mathbf{e}}_i) = \Gamma^k_{ji}\vec{\mathbf{e}}_k$$

   As the two covariant derivatives are equivalent, we are forced to conclude that:

$$\Gamma^k_{ij}\vec{\mathbf{e}}_k = \Gamma^k_{ji}\vec{\mathbf{e}}_k \tag{1}$$

2. Next, we establish metric compatibility.

$$\nabla_{\vec{\mathbf{e}}_k}(\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j) = (\nabla_{\vec{\mathbf{e}}_k}\vec{\mathbf{e}}_i) \cdot \vec{\mathbf{e}}_j + \vec{\mathbf{e}}_i \cdot (\nabla_{\vec{\mathbf{e}}_k \vec{\mathbf{e}}_j})$$

   This is given by the assumption that the dot product of two vectors stays the same when parallel transported along a curved plane, allowing the covariant derivative term to turn into a partial derivative as the dot product of any two vectors is a scalar.

$$\frac{\partial}{\partial \vec{\mathbf{e}}_k}(\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j) = (\Gamma^l_{ik}\vec{\mathbf{e}}_l) \cdot \vec{\mathbf{e}}_j + \vec{\mathbf{e}}_i \cdot (\Gamma^l_{kj}\vec{\mathbf{e}}_l)$$

   As defined in Table, the dot product of basis vectors gives the metric, so the equation simplifies to:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_k}(g_{ij}) = \Gamma^l_{ik}g_{lj} + \Gamma^l_{kj}g_{il} \tag{2}$$

3. Using the above two properties, we may create a definition of the connection coefficients

that is solely dependent on the metric and the basis vectors:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_k}(\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_j) = (\Gamma^l_{ik}\vec{\mathbf{e}}_l) \cdot \vec{\mathbf{e}}_j + \vec{\mathbf{e}}_i \cdot (\Gamma^l_{kj}\vec{\mathbf{e}}_l)$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_k}(g_{ij}) = \Gamma^l_{ik} g_{lj} + \Gamma^l_{kj} g_{il}$$
$$\frac{\partial}{\partial \vec{\mathbf{e}}_j}(\vec{\mathbf{e}}_k \cdot \vec{\mathbf{e}}_i) = (\Gamma^l_{jk}\vec{\mathbf{e}}_l) \cdot \vec{\mathbf{e}}_i + \vec{\mathbf{e}}_k \cdot (\Gamma^l_{ji}\vec{\mathbf{e}}_l)$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_j}(g_{ki}) = \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl}$$
$$\frac{\partial}{\partial \vec{\mathbf{e}}_i}(\vec{\mathbf{e}}_j \cdot \vec{\mathbf{e}}_k) = (\Gamma^l_{ij}\vec{\mathbf{e}}_l) \cdot \vec{\mathbf{e}}_k + \vec{\mathbf{e}}_j \cdot (\Gamma^l_{ik}\vec{\mathbf{e}}_l)$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_i}(g_{jk}) = \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl}$$

We simply add any two of the above terms together and subtract the third, resulting in:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_k}(g_{ij}) + \frac{\partial}{\partial \vec{\mathbf{e}}_j}(g_{ki}) - \frac{\partial}{\partial \vec{\mathbf{e}}_i}(g_{jk})$$
$$= \Gamma^l_{ik} g_{lj} + \Gamma^l_{kj} g_{il} + \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl}$$

Now, given the torsion free equation in (1), we combine like terms:

$$\Gamma^l_{ik} g_{lj} + \Gamma^l_{kj} g_{il} + \Gamma^l_{kj} g_{il} + \Gamma^l_{ij} g_{lk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{lj} = 2\Gamma^l_{kj} g_{il}$$

We multiply by $\frac{1}{2}$ and do a summation over the inverse metric tensor g^{mi} to get:

$$\frac{1}{2} g^{mi}(\frac{\partial}{\partial \vec{\mathbf{e}}_k}(g_{ij}) + \frac{\partial}{\partial \vec{\mathbf{e}}_j}(g_{ki}) - \frac{\partial}{\partial \vec{\mathbf{e}}_i}(g_{jk})) = g^{mi} g_{il} \Gamma^l_{kj}$$
$$= \delta^m_l \Gamma^l_{kj}$$
$$= \Gamma^m_{jk}$$

For the sake of consistency, we may generalize this definition to include the 3 space time dimensions with the Latin to Greek index substitutions:

$$m \mapsto \sigma$$
$$j \mapsto \mu$$
$$k \mapsto \nu$$
$$i \mapsto \alpha$$

We arrive at the generalized definition of the set of connection coefficients:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}(\frac{\partial}{\partial \vec{\mathbf{e}}_\nu} g_{\mu\alpha} + \frac{\partial}{\partial \vec{\mathbf{e}}_\mu} g_{\nu\alpha} - \frac{\partial}{\partial \vec{\mathbf{e}}_\alpha} g_{\mu\nu}) \quad (3)$$

The family of connection coefficients with this formula is known as the Levi-Civita Connection.

2 The Schwarzschild Metric Derivation

An extraordinarily influential solution of the Einstein Field Equations, the Schwarzschild metric proved to have applications spanning all stretches of modern physics, from cosmology to quantum mechanics. This solution of the E.F.E's makes starts with the initial assumptions of a spherically symmetric, non-rotating black hole that is static in time (allowing all time derivatives of the metric to go to 0) We begin with the Cartesian Minkowski Metric for flat space time:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

In spherical coordinates (ct, r, θ, ϕ), the same metric is represented as:

$$\begin{gathered} ct \mapsto ct \\ x \mapsto r\sin(\theta)\cos(\phi) \\ y \mapsto r\sin(\theta)\sin(\phi) \\ z \mapsto r\cos(\theta) \\ \Downarrow \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \end{gathered}$$

2.1 The Form of the Metric

We first establish the fact that, as r approaches infinitely far away from the mass, the metric must simplify to the flat Minkowski metric above. We also must come to the conclusion that the $g_{\theta\theta}$ and $g_{\phi\phi}$ components of the metric should be identical to those of a sphere of radius r (ignoring potential radial scaling). Maintaining spherical symmetry, the g_{tt} and g_{rr} of the metric must be solely dependant on r. From these assumptions, we end up with a metric of the form:

$$g_{\mu\nu} = \begin{bmatrix} f(r) & 0 & 0 & 0 \\ 0 & -h(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \tag{4}$$

We then establish the fact that we are working with a solution for the E.F.E's that applies to a body outside of the gravitational field of the earth, so the stress-energy-momentum tensor $T_{\mu\nu}$

approaches 0 and the equation simplifies:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$
$$\Rightarrow R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

We then do a summation over the inverse metric:

$$g^{\mu\nu}R_{\mu\nu} = \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu}$$
$$\Rightarrow R^{\mu}_{\mu} = \frac{1}{2}R\delta^{\mu}_{\mu}$$
$$\Rightarrow R = 2R$$

The only value for the Ricci scalar that satisfies the above equation is 0. Hence, the Ricci Tensor term is 0, meaning that volumes do not change along geodesics, and our metric is "Ricci Flat."

2.2 Calculating Connection Coefficients

Now, we may begin calculating our non-zero connection coefficients with the definition of the Levi-Civita Connection. Note that Latin indices (i, j, k) will be used to represent purely spatial basis vectors, whereas Greek indices (σ, μ, ν) can represent any one of the (ct, r, θ, ϕ) bases.

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}(\frac{\partial}{\partial\vec{\mathbf{e}}_{\nu}}g_{\mu\alpha} + \frac{\partial}{\partial\vec{\mathbf{e}}_{\mu}}g_{\nu\alpha} - \frac{\partial}{\partial\vec{\mathbf{e}}_{\alpha}}g_{\mu\nu})$$

Recall that the Schwarzschild metric is diagonal, so nonzero connection terms must have $\sigma = \alpha$:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\sigma}(\frac{\partial}{\partial\vec{\mathbf{e}}_{\nu}}g_{\mu\sigma} + \frac{\partial}{\partial\vec{\mathbf{e}}_{\mu}}g_{\nu\sigma} - \frac{\partial}{\partial\vec{\mathbf{e}}_{\sigma}}g_{\mu\nu})$$

Our calculation begins with the connection terms where $\sigma = 0$:

$$\Gamma^{0}_{\mu\nu} = \frac{1}{2}g^{00}(\frac{\partial}{\partial\vec{\mathbf{e}}_{\nu}}g_{\mu 0} + \frac{\partial}{\partial\vec{\mathbf{e}}_{\mu}}g_{\nu 0} - \frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{\mu\nu})$$

Given by the fact that derivatives of the metric with respect to the $\vec{\mathbf{e}}_{\mathbf{0}}$ basis vector go to 0, we may cancel any terms of the form $(\frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{\mu\nu})$. We are left with:

$$\Gamma^{0}_{00} = 0$$
$$\Gamma^{0}_{ii} = \frac{1}{2}g^{00}(\frac{\partial}{\partial\vec{\mathbf{e}}_{i}}g_{i0} + \frac{\partial}{\partial\vec{\mathbf{e}}_{i}}g_{i0} - \frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{ii}) = 0$$
$$\Gamma^{0}_{ij} = \frac{1}{2}g^{00}(\frac{\partial}{\partial\vec{\mathbf{e}}_{j}}g_{i0} + \frac{\partial}{\partial\vec{\mathbf{e}}_{i}}g_{j0} - \frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{ij}) = 0$$
$$\Gamma^{0}_{i0} = \frac{1}{2}g^{00}(\frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{i0} + \frac{\partial}{\partial\vec{\mathbf{e}}_{i}}g_{00} - \frac{\partial}{\partial\vec{\mathbf{e}}_{0}}g_{i0}) \neq 0 :$$

By the spherically symmetric property of the metric, we know the g_{00} component, or $f(r)$, is solely dependent on r. Thus, all terms besides those where $i = 1$ go to zero. We then use the metric established in (4) as the source of the metric components:

$$\Gamma^0_{10} = \frac{1}{2} g^{00} (\frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{10} + \frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{00} - \frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{10})$$
$$\Rightarrow \Gamma^0_{10} = \frac{1}{2f} (\frac{\partial f}{\partial \vec{\mathbf{e}}_r})$$

2.2.1 Case $\sigma = 1$

We use the process entailed above to find the rest of of the nonzero components. For components with $\sigma = 1$, we have:

$$\Gamma^1_{\mu\nu} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_\nu} g_{\mu 1} + \frac{\partial}{\partial \vec{\mathbf{e}}_\mu} g_{\nu 1} - \frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{\mu\nu})$$

We first calculate the connection coefficients for the case where μ and ν are the time-like coordinate:

$$\left.\begin{aligned} \Gamma^1_{00} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{01} + \frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{01} - \frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{00}) \\ \Rightarrow \Gamma^1_{00} = \frac{1}{2} g^{11} (-\frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{00}) \\ \Rightarrow \Gamma^1_{00} = -\frac{1}{2h} (-\frac{\partial f}{\partial \vec{\mathbf{e}}_r}) \end{aligned}\right\} (\mu = 0,\ \nu = 0)$$

We then calculate the connection coefficients for the case where μ and ν are the same space-like coordinate:

$$\left.\begin{aligned} \Gamma^1_{ii} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_i} g_{i1} + \frac{\partial}{\partial \vec{\mathbf{e}}_i} g_{i1} - \frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{ii}) \\ \\ \Gamma^1_{11} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{11}) \\ \Rightarrow \Gamma^1_{11} = -\frac{1}{2h} (-\frac{\partial h}{\partial \vec{\mathbf{e}}_r}) \\ \\ \Gamma^1_{22} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{22}) = -\frac{1}{2h} (-\frac{\partial r^2}{\partial \vec{\mathbf{e}}_r}) \\ \Rightarrow \Gamma^1_{22} = -\frac{r}{h} \\ \\ \Gamma^1_{33} = \frac{1}{2} g^{11} (\frac{\partial}{\partial \vec{\mathbf{e}}_1} g_{33}) = -\frac{1}{2h} (-\frac{\partial r^2 \sin^2(\theta)}{\partial \vec{\mathbf{e}}_r}) \\ \Rightarrow \Gamma^1_{33} = -\frac{r \sin^2(\theta)}{h} \end{aligned}\right\} (\mu = i,\ \nu = i)$$

Next, for the case where μ and ν are different space-like coordinates:

$$\left.\begin{aligned}
\Gamma^1_{ij} &= \frac{1}{2}g^{11}(\frac{\partial}{\partial\vec{\mathbf{e}}_j}g_{i1} + \frac{\partial}{\partial\vec{\mathbf{e}}_i}g_{j1} - \frac{\partial}{\partial\vec{\mathbf{e}}_1}g_{ij}) \\
\Gamma^1_{21} &= \frac{1}{2}g^{11}(\frac{\partial}{\partial\vec{\mathbf{e}}_2}g_{11}) = -\frac{1}{2h}(-\frac{\partial h}{\partial\vec{\mathbf{e}}_\theta}) \\
&\Rightarrow \Gamma^1_{21} = -\frac{1}{2h}(0) = 0 \\
\Gamma^1_{31} &= \frac{1}{2}g^{11}(\frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{11}) = -\frac{1}{2h}(-\frac{\partial h}{\partial\vec{\mathbf{e}}_\phi}) \\
&\Rightarrow \Gamma^1_{31} = -\frac{1}{2h}(0) = 0 \\
\Gamma^1_{32} &= \frac{1}{2}g^{11}(0) \\
&\Rightarrow \Gamma^1_{32} = 0
\end{aligned}\right\} (\mu = i,\ \nu = j)$$

And finally, the case where μ is a space-like coordinate and ν is the time coordinate:

$$\left.\begin{aligned}
\Gamma^1_{i0} &= \frac{1}{2}g^{11}(\frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{i1} + \frac{\partial}{\partial\vec{\mathbf{e}}_i}g_{01} - \frac{\partial}{\partial\vec{\mathbf{e}}_1}g_{i0}) \\
&\Rightarrow \frac{1}{2}g^{11}(0+0+0) \\
&\Rightarrow 0
\end{aligned}\right\} (\mu = i,\ \nu = 0)$$

2.2.2 Case $\sigma = 2$

For components with $\sigma = 2$, we have:

$$\Gamma^2_{\mu\nu} = \frac{1}{2}g^{22}(\frac{\partial}{\partial\vec{\mathbf{e}}_\nu}g_{\mu 2} + \frac{\partial}{\partial\vec{\mathbf{e}}_\mu}g_{\nu 2} - \frac{\partial}{\partial\vec{\mathbf{e}}_2}g_{\mu\nu})$$

We first calculate the connection coefficients for the case where μ and ν are the time-like coordinate:

$$\left.\begin{aligned}
\Gamma^2_{00} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{02} + \frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{02} - \frac{\partial}{\partial\vec{\mathbf{e}}_2}g_{00}) \\
&\Rightarrow \Gamma^2_{00} = \frac{1}{2}g^{22}(0) \\
&\Rightarrow \Gamma^2_{00} = 0
\end{aligned}\right\} (\mu = 0,\ \nu = 0)$$

We then calculate the connection coefficients for the case where μ and ν are the same space-like coordinate:

$$\left.\begin{aligned}
\Gamma^2_{ii} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_i}g_{i2} + \frac{\partial}{\partial \vec{\mathbf{e}}_i}g_{i2} - \frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{ii}) \\
\Gamma^2_{11} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{11}) \\
\Rightarrow \Gamma^2_{11} &= -\frac{1}{2r^2}(0) = 0 \\
\Gamma^2_{22} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{22}) = -\frac{1}{2r^2}(-\frac{\partial r^2}{\partial \vec{\mathbf{e}}_\theta}) \\
\Rightarrow \Gamma^2_{22} &= -\frac{1}{2r^2}(0) = 0 \\
\Gamma^2_{33} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{33}) = -\frac{1}{2r^2 \sin^2(\theta)}(-\frac{\partial r^2 \sin^2(\theta)}{\partial \vec{\mathbf{e}}_\theta}) \\
\Rightarrow \Gamma^2_{33} &= -\frac{2r^2 \sin\theta \cos\theta}{2r^2} = -\cos\theta \sin\theta
\end{aligned}\right\} (\mu = i,\ \nu = i)$$

Next, for the case where μ and ν are different space-like coordinates:

$$\left.\begin{aligned}
\Gamma^2_{ij} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_j}g_{i2} + \frac{\partial}{\partial \vec{\mathbf{e}}_i}g_{j2} - \frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{ij}) \\
\Gamma^2_{21} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_1}g_{22}) = -\frac{1}{2r^2}(-\frac{\partial r^2}{\partial \vec{\mathbf{e}}_r}) \\
\Rightarrow \Gamma^2_{21} &= -\frac{-2r}{2r^2} = \frac{1}{r} \\
\Gamma^2_{31} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_1}g_{32}) \\
\Rightarrow \Gamma^2_{31} &= -\frac{1}{2r^2}(0) = 0 \\
\Gamma^2_{32} &= \frac{1}{2}g^{22}(\frac{\partial}{\partial \vec{\mathbf{e}}_3}g_{22}) \\
\Rightarrow \Gamma^2_{32} &= -\frac{1}{2r^2}(0) = 0
\end{aligned}\right\} (\mu = i,\ \nu = j)$$

And finally, the case where μ is a space-like coordinate and ν is the time coordinate:

$$\left.\begin{aligned}\Gamma^2_{i0} = \frac{1}{2}g^{22}(\frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{i2} + \frac{\partial}{\partial\vec{\mathbf{e}}_i}g_{02} - \frac{\partial}{\partial\vec{\mathbf{e}}_2}g_{i0})\\ \Rightarrow \frac{1}{2}g^{22}(0+0-0)\\ \Rightarrow 0\end{aligned}\right\}(\mu = i,\ \nu = 0)$$

2.2.3 Case $\sigma = 3$

For components with $\sigma = 3$, we have:

$$\Gamma^3_{\mu\nu} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_\nu}g_{\mu 3} + \frac{\partial}{\partial\vec{\mathbf{e}}_\mu}g_{\nu 3} - \frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{\mu\nu})$$

We first calculate the connection coefficients for the case where μ and ν are the time-like coordinate:

$$\left.\begin{aligned}\Gamma^3_{00} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{03} + \frac{\partial}{\partial\vec{\mathbf{e}}_0}g_{03} - \frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{00})\\ \Rightarrow \Gamma^3_{00} = \frac{1}{2}g^{33}(0)\\ \Rightarrow \Gamma^3_{00} = 0\end{aligned}\right\}(\mu = 0,\ \nu = 0)$$

We then calculate the connection coefficients for the case where μ and ν are the same space-like coordinate:

$$\left.\begin{aligned}\Gamma^3_{ii} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_i}g_{i3} + \frac{\partial}{\partial\vec{\mathbf{e}}_i}g_{i3} - \frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{ii})\\ \\ \Gamma^3_{11} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{11})\\ \Rightarrow \Gamma^3_{11} = -\frac{1}{2r^2\sin^2(\theta)}(0) = 0\\ \\ \Gamma^3_{22} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{33}) = -\frac{1}{2r^2\sin^2(\theta)}(-\frac{\partial r^2}{\partial\vec{\mathbf{e}}_\phi})\\ \Rightarrow \Gamma^3_{22} = -\frac{1}{2r^2\sin^2(\theta)}(0) = 0\\ \\ \Gamma^3_{33} = \frac{1}{2}g^{33}(\frac{\partial}{\partial\vec{\mathbf{e}}_3}g_{33}) = -\frac{1}{2r^2\sin^2(\theta)}(-\frac{\partial r^2\sin^2(\theta)}{\partial\vec{\mathbf{e}}_\phi})\\ \Rightarrow \Gamma^3_{33} = 0\end{aligned}\right\}(\mu = i,\ \nu = i)$$

Next, for the case where μ and ν are different space-like coordinates:

$$\left.\begin{aligned}
\Gamma^3_{ij} = \frac{1}{2}g^{33}(\frac{\partial}{\partial \vec{\mathbf{e}}_j}g_{i3} + \frac{\partial}{\partial \vec{\mathbf{e}}_i}g_{j3} - \frac{\partial}{\partial \vec{\mathbf{e}}_3}g_{ij})\\
\\
\Gamma^3_{21} = \frac{1}{2}g^{33}(\frac{\partial}{\partial \vec{\mathbf{e}}_1}g_{33}) = -\frac{1}{2r^2\sin^2(\theta)}(0)\\
\Rightarrow \Gamma^3_{21} = 0\\
\\
\Gamma^3_{31} = \frac{1}{2}g^{33}(\frac{\partial}{\partial \vec{\mathbf{e}}_1}g_{33})\\
\Rightarrow \Gamma^3_{31} = \frac{1}{2r^2\sin^2(\theta)}(2r\sin^2(\theta)) = \frac{1}{r}\\
\\
\Gamma^3_{32} = \frac{1}{2}g^{33}(\frac{\partial}{\partial \vec{\mathbf{e}}_2}g_{33})\\
\Rightarrow \Gamma^3_{32} = \frac{1}{2r^2\sin^2(\theta)}(2r^2\sin(\theta)\cos(\theta))\\
\Rightarrow \Gamma^3_{32} = \cot(\theta)
\end{aligned}\right\}(\mu = i,\ \nu = j)$$

And finally, the case where μ is a space-like coordinate and ν is the time coordinate:

$$\left.\begin{aligned}
\Gamma^3_{i0} = \frac{1}{2}g^{33}(\frac{\partial}{\partial \vec{\mathbf{e}}_0}g_{i3} + \frac{\partial}{\partial \vec{\mathbf{e}}_i}g_{03} - \frac{\partial}{\partial \vec{\mathbf{e}}_3}g_{i0})\\
\Rightarrow \frac{1}{2}g^{33}(0 + 0 - 0)\\
\Rightarrow \Gamma^3_{i0} = 0
\end{aligned}\right\}(\mu = i,\ \nu = 0)$$

Thus, we may conclude that our only nonzero connection coefficients are as follows:

Nonzero Connection Coefficients
$\Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{2f}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}$
$\Gamma^1_{00} = \frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}$
$\Gamma^1_{11} = \frac{1}{2h}\frac{\partial h}{\partial \vec{\mathbf{e}}_r}$
$\Gamma^1_{22} = -\frac{r}{h}$
$\Gamma^1_{33} = -\frac{r\sin^2(\theta)}{h}$
$\Gamma^2_{33} = -\cos\theta\sin\theta$
$\Gamma^2_{21} = \Gamma^2_{12} = \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r}$
$\Gamma^3_{32} = \Gamma^3_{23} = \cot(\theta)$

2.3 Components of the Ricci Curvature Tensor

Recall that the Ricci Curvature Tensor components are necessary for solving the E.F.E.'s, so we must calculate them. We begin with the Riemann Curvature Tensor equations:

$$R(\vec{\mathbf{e}}_\mu, \vec{\mathbf{e}}_\nu)\vec{\mathbf{e}}_\sigma = \nabla_{\vec{\mathbf{e}}_\mu}\nabla_{\vec{\mathbf{e}}_\nu}\vec{\mathbf{e}}_\sigma - \nabla_{\vec{\mathbf{e}}_\nu}\nabla_{\vec{\mathbf{e}}_\mu}\vec{\mathbf{e}}_\sigma - \nabla_{[\vec{\mathbf{e}}_\mu,\vec{\mathbf{e}}_\nu]}\vec{\mathbf{e}}_\sigma$$
$$R^\rho_{\sigma\mu\nu} = \frac{\partial}{\partial \vec{\mathbf{e}}_\mu}\Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\sigma\nu} - \frac{\partial}{\partial \vec{\mathbf{e}}_\nu}\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\beta\nu}\Gamma^\beta_{\mu\sigma}$$

The Ricci Curvature Tensor components are given by simple contraction with the metric tensor over the μ index:

$$R_{\sigma\nu} = g_{\rho\mu}R^\rho_{\sigma\mu\nu} = R^\mu_{\rho\mu\nu}$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_\mu}\Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\mu\alpha}\Gamma^\alpha_{\sigma\nu} - \frac{\partial}{\partial \vec{\mathbf{e}}_\nu}\Gamma^\mu_{\mu\sigma} - \Gamma^\mu_{\beta\nu}\Gamma^\beta_{\mu\sigma}$$

We may now begin our calculation of the tensor components.

2.3.1 Component R_{00}

$$R_{00} = g_{\rho\mu}R^\rho_{0\mu0} = R^\mu_{0\mu0}$$
$$\Rightarrow R^\mu_{0\mu0} = \frac{\partial}{\partial \vec{\mathbf{e}}_\mu}\Gamma^\mu_{00} + \Gamma^\mu_{\mu\alpha}\Gamma^\alpha_{00} - \frac{\partial}{\partial \vec{\mathbf{e}}_0}\Gamma^\mu_{\mu0} - \Gamma^\mu_{\beta0}\Gamma^\beta_{\mu0}$$

We may now expand our summations in terms of the nonzero connection coefficients, sending any time-derivatives to 0:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_1}\Gamma^1_{00} + (\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{00} - \Gamma^0_{10}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{10}$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_r}\left(\frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}\right) + \frac{1}{2h}\frac{\partial h}{\partial \vec{\mathbf{e}}_r}\frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r} + 2\frac{1}{r}(\frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}) - \frac{1}{2f}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}\frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}$$
$$\Rightarrow \frac{\partial}{\partial \vec{\mathbf{e}}_r}\left(\frac{1}{2h}\frac{\partial f}{\partial \vec{\mathbf{e}}_r}\right) + \frac{1}{4h^2}\frac{\partial h}{\partial \vec{\mathbf{e}}_r}\frac{\partial f}{\partial \vec{\mathbf{e}}_r} + \frac{1}{hr}\frac{\partial f}{\partial \vec{\mathbf{e}}_r} - \frac{1}{4fh}(\frac{\partial f}{\partial \vec{\mathbf{e}}_r})^2$$

Expanding the first term with single-variable chain rule, and simplifying notation:

$$\frac{f''}{2h} - \frac{f'h'}{2h^2} + \frac{f'h'}{4h^2} + \frac{f'}{hr} - \frac{(f')^2}{4fh} = 0$$

Simplifying further and eliminating the denominator:

$$\Rightarrow \left(\frac{f''}{2h} - \frac{f'h'}{4h^2} + \frac{f'}{hr} - \frac{(f')^2}{4fh}\right)(4h^2fr) = 0$$
$$\Rightarrow R_{00} = 2f''fhr - f'fh'r + 4f'fh - (f')^2hr = 0$$

2.3.2 Component R_{11}

$$R_{11} = g_{\rho\mu}R^{\rho}_{1\mu 1} = R^{\mu}_{1\mu 1}$$
$$\Rightarrow R^{\mu}_{1\mu 1} = \frac{\partial}{\partial \vec{\mathbf{e}}_\mu}\Gamma^{\mu}_{11} + \Gamma^{\mu}_{\mu\alpha}\Gamma^{\alpha}_{11} - \frac{\partial}{\partial \vec{\mathbf{e}}_1}\Gamma^{\mu}_{\mu 1} - \Gamma^{\mu}_{\beta 1}\Gamma^{\beta}_{\mu 1}$$

Expanding the summations:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_1}\Gamma^1_{11} + (\Gamma^0_{01} + \Gamma^1_{11} + 2\Gamma^2_{21})\Gamma^1_{11} - \frac{\partial}{\partial \vec{\mathbf{e}}_1}(\Gamma^0_{01} + \Gamma^1_{11} + 2\Gamma^2_{21}) - (\Gamma^0_{01}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^1_{11} + 2\Gamma^2_{21}\Gamma^2_{12})$$
$$\Rightarrow (\Gamma^0_{01} + 2\Gamma^2_{21})\Gamma^1_{11} - \frac{\partial}{\partial \vec{\mathbf{e}}_1}(\Gamma^0_{01} + 2\Gamma^2_{21}) - (\Gamma^0_{01}\Gamma^0_{10} + 2\Gamma^2_{21}\Gamma^2_{12})$$
$$\Rightarrow \frac{f'}{2f}\frac{h'}{2h} + \frac{2}{r}\frac{h'}{2h} - \frac{\partial}{\partial \vec{\mathbf{e}}_r}\left(\frac{f'}{2f} + \frac{2}{r}\right) - \left(\frac{f'}{2f}\right)^2 - 2\left(\frac{1}{r^2}\right)$$

Simplifying further:

$$\frac{f'h'}{4fh} + \frac{h'}{hr} - \frac{\partial}{\partial \vec{\mathbf{e}}_r}\left(\frac{f'}{2f} + \frac{2}{r}\right) - \left(\frac{f'}{2f}\right)^2 - \left(\frac{2}{r^2}\right)$$

Simplifying further and eliminating the denominator:

$$\left(\frac{f'h'}{4fh} + \frac{h'}{hr} - \frac{f''}{2f} + \frac{(f')^2}{4f^2}\right)(4hf^2r) = 0$$
$$\Rightarrow R_{11} = f'fh'r + 4f^2h' - 2f''fhr + (f')^2hr = 0$$

2.3.3 Component R_{22}

$$R_{22} = g_{\rho\mu} R^{\rho}_{2\mu 2} = R^{\mu}_{2\mu 2}$$
$$\Rightarrow R^{\mu}_{2\mu 2} = \frac{\partial}{\partial \vec{\mathbf{e}}_{\mu}} \Gamma^{\mu}_{22} + \Gamma^{\mu}_{\mu\alpha}\Gamma^{\alpha}_{22} - \frac{\partial}{\partial \vec{\mathbf{e}}_{2}} \Gamma^{\mu}_{\mu 2} - \Gamma^{\mu}_{\beta 2}\Gamma^{\beta}_{\mu 2}$$

Expanding the summations:

$$\frac{\partial}{\partial \vec{\mathbf{e}}_1}\Gamma^1_{22} + (\Gamma^0_{01} + \Gamma^1_{11} + 2\Gamma^2_{21})\Gamma^1_{22} - \frac{\partial}{\partial \vec{\mathbf{e}}_2}(\Gamma^3_{32}) - (2\Gamma^2_{21}\Gamma^1_{22} + \Gamma^3_{32}\Gamma^3_{32})$$
$$\Rightarrow -\frac{\partial}{\partial \vec{\mathbf{e}}_r}\left(\frac{r}{h}\right) - \left(\frac{f'}{2f} + \frac{h'}{2h}\right)\left(\frac{r}{h}\right) - \frac{\partial}{\partial \vec{\mathbf{e}}_\theta}(\cot\theta) - (\cot\theta)(\cot\theta)$$
$$\Rightarrow -\frac{1}{h} + \frac{rh'}{h^2} - \frac{f'r}{2fh} - \frac{h'r}{2h^2} - (-\csc^2\theta + \cot^2\theta)$$
$$\Rightarrow -\frac{1}{h} - \frac{f'r}{2fh} + \frac{h'r}{2h^2} + 1$$

Simplifying further and eliminating the denominator:

$$\left(-\frac{1}{h} - \frac{f'r}{2fh} + \frac{h'r}{2h^2} + 1\right)(2fh^2)$$
$$\Rightarrow R_{22} = -2fh + fh'r - f'hr + 2fh^2 = 0$$

2.4 Solving for $f(r)$ and $h(r)$

As each of the above components is equal to 0, their sum must also be 0. Using this fact, we are able to solve for f and h as functions of r.

$$R_{00} = 2f''fhr - f'fh'r + 4f'fhr - (f')^2hr = 0$$
$$R_{11} = f'fh'r + 4f^2h' - 2f''fhr + (f')^2hr = 0$$

We may then infer that $R_{00} + R_{11} = 0$

$$R_{00} + R_{11} = 0$$
$$2f''fhr - f'fh'r + 4f'fhr - (f')^2hr + f'fh'r + 4f^2h' - 2f''fhr + (f')^2hr = 0$$
$$\Rightarrow 4f'fhr + 4f^2h'\left(\frac{1}{4f}\right) = 0\left(\frac{1}{4f}\right)$$
$$\Rightarrow f'h + fh' = 0$$

Rudimentary calculus will tell us that this equation is simply the product rule, and can be simplified to:

$$\frac{\partial(fh)}{\partial \vec{\mathbf{e}}_r} = 0$$

This implies that $f(r)h(r)$ is constant:

$$f(r)h(r) = K$$

As r approaches infinity, any $f(r)$ approaches 1 (given by the definition of the Schwarzschild metric):

$$\lim_{r\to\infty} f(r)h(r) = 1 = K$$
$$h(r) = \frac{1}{f(r)}$$

This relationship holds true for all r. We may apply this definition to our equation for the R_{22} component of the Ricci curvature tensor:

$$h = \frac{1}{f}$$
$$R_{22} = -2fh + fh'r - f'hr + 2fh^2 = 0$$
$$\Rightarrow R_{22} = -2f\left(\frac{1}{f}\right) + f\left(\frac{-f'}{f^2}\right)r - f'\left(\frac{1}{f}\right)r + 2f\left(\frac{1}{f^2}\right) = 0$$
$$\Rightarrow -2 - \frac{f'r}{f} - \frac{f'}{f}r + \frac{2}{f} = 0$$
$$\Rightarrow -2f - 2f'r + 2 = 0 \Rightarrow f'r = 1 - f$$

We now solve this differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{1-f}{r}$$
$$\Rightarrow \frac{\mathrm{d}f}{1-f} = \frac{\mathrm{d}r}{r}$$
$$\Rightarrow \int \frac{1}{1-f}\,df = \int \frac{1}{r}\,dr$$
$$\Rightarrow -\ln(1-f) = \ln(r) + c$$
$$\Rightarrow e^{\ln(1-f)^{-1}} = e^{\ln(r)+c}$$
$$\Rightarrow 1 - \frac{1}{f} = Cr$$
$$\Rightarrow f(r) = \frac{1}{1-Cr} \Rightarrow 1 - \frac{1}{Cr}$$

Setting $k \equiv C^{-1}$, we are left with the following definitions:

$$f(r) = 1 - \frac{k}{r}$$
$$h(r) = (1 - \frac{k}{r})^{-1} \tag{5}$$

Now, to solve for K, we force our metric to match that of Newtonian gravity upon application of the weak field and low velocity limits. This process includes the following mathematical assertions:

$$\frac{\mathrm{d}x^\sigma}{\mathrm{d}t} = \vec{\mathbf{U}} = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7}$$

$$\begin{aligned} m\vec{\mathbf{g}} = \frac{GmM}{r^2} &\Rightarrow \vec{\mathbf{g}} = \frac{GM}{r^2} \\ \vec{\mathbf{g}} = \frac{GM}{r^2} &= -\nabla(V) = -\frac{\partial V}{\partial \vec{\mathbf{e}}_i} \\ &\Rightarrow \frac{\partial V}{\partial \vec{\mathbf{e}}_r} = -\frac{GM}{r^2} \\ &\Rightarrow V(r) = \int -\frac{GM}{r^2}\, dr \\ &\Rightarrow V(r) = -\frac{GM}{r} \end{aligned} \tag{8}$$

Beginning with a brief derivation of the geodesic equation for Newtonian gravity:

$$\vec{\mathbf{a}} = \vec{\mathbf{g}}$$

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \nabla V = 0$$

$$\Rightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\partial V}{\partial x^i} = 0$$

We then perform the same operation with our low-velocity limit bounds:

$$\frac{\mathrm{d}x^\sigma}{\mathrm{d}t} = \vec{\mathbf{U}} = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\Gamma^\sigma_{ij} = 0$$

$$\Rightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\mathrm{d}x^0}{\mathrm{d}t}\frac{\mathrm{d}x^0}{\mathrm{d}t}\Gamma^\sigma_{00}$$

$$\Rightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Gamma^\sigma_{ii}$$

$$\Rightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + c^2\Gamma^i_{00} = 0$$

If the metric is to match Newtonian gravity at the low velocity limit:

$$\frac{\partial V}{\partial x^i}\left(\frac{1}{c^2}\right) \equiv \Gamma^i_{00}$$

Finally, we apply the weak field limit to the definition of the connection:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial}{\partial \vec{\mathbf{e}}_\nu} g_{\mu\alpha} + \frac{\partial}{\partial \vec{\mathbf{e}}_\mu} g_{\nu\alpha} - \frac{\partial}{\partial \vec{\mathbf{e}}_\alpha} g_{\mu\nu} \right)$$
$$\Rightarrow \Gamma^{i}_{00} = \frac{1}{2} g^{ii} \left(\frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{0i} + \frac{\partial}{\partial \vec{\mathbf{e}}_0} g_{0i} - \frac{\partial}{\partial \vec{\mathbf{e}}_i} g_{00} \right)$$
$$\Rightarrow \Gamma^{i}_{00} = \frac{1}{2} \eta^{ii} \left(\frac{\partial}{\partial \vec{\mathbf{e}}_0} h_{0i} + \frac{\partial}{\partial \vec{\mathbf{e}}_0} h_{0i} - \frac{\partial}{\partial \vec{\mathbf{e}}_i} h_{00} \right)$$

Sending any off-diagonal terms of the metric to 0 and the Minkowski metric term to -1:

$$\Gamma^{i}_{00} \equiv \frac{1}{2} \left(\frac{\partial}{\partial \vec{\mathbf{e}}_i} h_{00} \right)$$

We may set our two definitions of the connection equal to each other to solve for the h_{00} components of the metric:

$$\Gamma^{i}_{00} = \frac{1}{2} \left(\frac{\partial}{\partial \vec{\mathbf{e}}_i} h_{00} \right) = \frac{\partial V}{\partial x^i} \left(\frac{1}{c^2} \right)$$
$$\Rightarrow \frac{\partial h_{00}}{\partial \vec{\mathbf{e}}_i} = \frac{\partial V}{\partial \vec{\mathbf{e}}_i} \left(\frac{2}{c^2} \right)$$

Substituting V for the function found in 8:

$$\frac{\partial h_{00}}{\partial \vec{\mathbf{e}}_i} = \frac{\partial V}{\partial \vec{\mathbf{e}}_i} \left(\frac{2}{c^2} \right)$$
$$\frac{\partial h_{00}}{\partial \vec{\mathbf{e}}_r} = \frac{2}{c^2} \left(\frac{GM}{r^2} \right) = \left(\frac{2GM}{r^2 c^2} \right) = \left(\frac{2GM}{c^2} \right) \frac{1}{r^2}$$
$$-h_{00} = \left[-\frac{2GM}{c^2 r} \right]_{r=r}^{r=\infty}$$
$$\Rightarrow h_{00} = -\frac{2GM}{c^2 r}$$

Applying both (7) and (5):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow g_{00} = \eta_{00} + h_{00}$$
$$g_{00} = 1 - \frac{2GM}{c^2 r}$$
$$g_{00} = 1 - \frac{k}{r} \Rightarrow k = \frac{2GM}{c^2}$$

This "k" constant is marked as the Schwarzschild Radius, and with this, our derivation of the metric is complete:

$$\begin{bmatrix} 1-\frac{2GM}{c^2r} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2GM}{c^2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \tag{9}$$

Oftentimes, we refer to this $\frac{2GM}{c^2}$ term as the **Schwarzschild Radius**, denoted by the symbol r_s:

$$\begin{bmatrix} 1-\frac{r_s}{r} & 0 & 0 & 0 \\ 0 & \left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \tag{10}$$

3 Analysis of the Schwarzschild Metric

As the Schwarzschild metric matches that of the metric for an arbitrary, spherically symmetric mass in the $\vec{\mathbf{e}}_\theta$ and $\vec{\mathbf{e}}_\phi$ directions, our primary focus will be on the bases on which it differs from traditional spherical symmetry, namely $\vec{\mathbf{e}}_t$ and $\vec{\mathbf{e}}_r$. We begin by considering the squared length (spacetime interval) of the $\vec{\mathbf{e}}_t$ and $\vec{\mathbf{e}}_r$ basis vectors. Recall that light-like (null) geodesics have a squared spacetime interval of 0.

$$s^2 = \left|\left|\frac{\partial}{\partial\lambda}\right|\right| = \left|\frac{\partial}{\partial\lambda}\cdot\frac{\partial}{\partial\lambda}\right|$$

As we are assuming radial motion only, we may express λ solely in terms of the $\vec{\mathbf{e}}_t$ and $\vec{\mathbf{e}}_r$ basis vectors.

$$\Rightarrow \left|\left(\frac{\partial ct}{\partial\lambda}\frac{\partial}{\partial ct}+\frac{\partial r}{\partial\lambda}\frac{\partial}{\partial r}\right)\cdot\left(\frac{\partial ct}{\partial\lambda}\frac{\partial}{\partial ct}+\frac{\partial r}{\partial\lambda}\frac{\partial}{\partial r}\right)\right| = 0$$

Recall that all off-diagonal components of the metric go to 0, so all non-like basis vector dot products must go to 0 and we are left with:

$$\Rightarrow \left(\frac{\partial ct}{\partial\lambda}\right)^2 g_{tt} + \left(\frac{\partial r}{\partial\lambda}\right)^2 g_{rr} = 0$$
$$\Rightarrow \left(\frac{\partial ct}{\partial\lambda}\right)^2\left(\frac{r-r_s}{r}\right) - \left(\frac{\partial r}{\partial\lambda}\right)^2\left(\frac{r}{r-r_s}\right) = 0$$
$$\Rightarrow \left(\frac{\partial ct}{\partial\lambda}\right)^2\left(\frac{r-r_s}{r}\right) = \left(\frac{\partial r}{\partial\lambda}\right)^2\left(\frac{r}{r-r_s}\right)$$
$$\Rightarrow \left(\frac{\partial ct}{\partial\lambda}\right)^2\left(\frac{r-r_s}{r}\right)^2 = \left(\frac{\partial r}{\partial\lambda}\right)^2$$

In order to find curves of **ct** traveling along **r**, we set our arbitrary path parameter λ to **r**:

$$\Rightarrow \left(\frac{\partial ct}{\partial r}\right)^2 \left(\frac{r - r_s}{r}\right)^2 = \left(\frac{\partial r}{\partial r}\right)^2$$
$$= 1$$
$$\Rightarrow \left(\frac{\partial ct}{\partial r}\right)^2 = \left(\frac{r}{r - r_s}\right)^2$$

We are left with the following differential equation:

$$\frac{\partial ct}{\partial r} = \pm \left(\frac{r}{r - r_s}\right)$$

Integrating by parts, we are left with:

$$ct(r) = \pm(r + r_s \ln{(r - r_s)} + k)$$

For the sake of consistency in notation (in order for the equation of null geodesics to more closely match the notation of the metric, we express the above equation in the equivalent form:

$$ct(r) = \pm \left(r + r_s \ln\left(\frac{r - r_s}{r_s}\right) + C\right)$$

Although this may seem counter intuitive, this mathematical trick does not change any values relevant to the differential equation, and this is still a valid solution. Also note the change in denotation of the arbitrary constant k to C, as this value does change. The reader is invited to take the derivative of our new expression as a sanity check, but going forward, the below notation will be referenced often.

$$ct(r) = \pm \left(r + r_s \ln\left(\frac{r}{r_s} - 1\right)\right) + C$$

This equation makes the assumption that our light beams originate outside the Schwarzschild radius ($r > r_s$). If it were not, our ln term would be undefined. To account for light beams inside the Schwarzschild radius ($r < r_s$), only slight modification is needed. The $\pm$ sign is also defined based on whether our light beams are facing inward (in-going) or outward (outgoing). With the above information, we arrive at the following equations for geodesics.

For $r > r_s$

In-going:

$$ct(r) = -r - r_s \ln\left(\frac{r}{r_s} - 1\right) + C \tag{11}$$

Outgoing:

$$ct(r) = r + r_s \ln\left(\frac{r}{r_s} - 1\right) + C \tag{12}$$

For $r < r_s$

In-going:

$$ct(r) = -r - r_s \ln\left(1 - \frac{r}{r_s}\right) + C \tag{13}$$

Outgoing:

$$ct(r) = r + r_s \ln\left(1 - \frac{r}{r_s}\right) + C \tag{14}$$

3.1 Eddington-Finklestein Coordinates

Immediately, a number of issues with our current version of the metric become apparent. To start, the metric itself lacks formal definition at the Schwarzschild radius $r = r_s$, effectively breaking our coordinate system. Additionally, with our current equation for light-like geodesics appears to be disconnected at the Schwarzschild radius. Our first step in rectifying the above issues should be converting our current coordinate system to one which treats null geodesics as lines of constant coordinate.

We begin with our current equation for in-going light-like geodesics:

$$ct(r) = -r - r_s \ln\left|\frac{r}{r_s} - 1\right| + C$$

We can define a new coordinate **v** such that:

$$ct \to v$$
$$v \equiv ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|$$

Our new in-going null geodesics in terms of **v** are simply given by:

$$ct(r) = -r - r_s \ln\left|\frac{r}{r_s} - 1\right| + C$$
$$v = r + r_s \ln\left|\frac{r}{r_s} - 1\right| - r - r_s \ln\left|\frac{r}{r_s} - 1\right| + C$$
$$\Rightarrow v = C$$

Expressing our outgoing geodesics in terms of **v**:

$$ct(r) = r + r_s \ln\left|\frac{r}{r_s} - 1\right| + C$$
$$v = ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right| = 2r + 2r_s \ln\left|\frac{r}{r_s} - 1\right| + C$$

Clearly in-going null geodesics travel along curves of constant **v** whereas outgoing null geodesics do not. This new coordinate system is known as in-going Turtle coordinates (or in-going Eddington-Finklestein coordinates), as its in-going light-like geodesics appear to travel along constant coordinate curves. To clarify, our new coordinates are:

$$ct \to v$$
$$r \to r_i$$

Whose transformation rules are given by:

$$v \equiv ct + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$
$$r_i \equiv r \tag{15}$$

Now, we repeat the above procedure for null geodesics at $r < r_s$

Beginning with our current equation for in-going light-like geodesics:

$$ct(r) = -r - r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$

We define a new coordinate **u** such that:

$$ct \to u$$
$$u \equiv ct - r - r_s \ln\left|1 - \frac{r}{r_s}\right|$$

Our new in-going null geodesics in terms of **u**:

$$ct(r) = -r - r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$
$$u = -r - r_s \ln\left|1 - \frac{r}{r_s}\right| - r - r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$
$$\Rightarrow u = -2r - 2r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$

Expressing our outgoing geodesics in terms of **u**:

$$ct(r) = r + r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$
$$u = -r - r_s \ln\left|1 - \frac{r}{r_s}\right| + r + r_s \ln\left|1 - \frac{r}{r_s}\right| + C$$
$$\Rightarrow u = C$$

Outgoing null geodesics travel along curves of constant u, whereas outgoing null geodesics do not. This is known as the Turtle (or outgoing Eddington-Finklestein) coordinate system as its outgoing light-like geodesics travel along constant coordinate curves. This set of new coordinates are:

$$ct \to u$$
$$r \to r_o$$

These transformation rules are given by:

$$u \equiv ct - r - r_s \ln\left|1 - \frac{r}{r_s}\right| \qquad (16)$$
$$r_o \equiv r$$

3.1.1 In-going EF Metric

We have now constructed two new sets of coordinates, each of which can be used to express the Schwarzschild metric. Naturally, our next step is to find these metric tensor components. As the metric tensor is composed of individual basis vector dot products, we must begin by finding the basis vectors in terms of partial derivative operators. Beginning with the in-going Eddington-Finklestein coordinates **v** and $\mathbf{r_i}$:

$$\frac{\partial}{\partial v} = \left(\frac{\partial ct}{\partial v}\frac{\partial}{\partial ct} + \frac{\partial r}{\partial v}\frac{\partial}{\partial r}\right)$$

We may expand the above expression in terms of the transformation rules found above:

$$\Rightarrow \frac{\partial}{\partial v} = 1\frac{\partial}{\partial ct} + 0\frac{\partial}{\partial r}$$

The aforementioned transformation rules also allow us to find an expression for $\mathbf{r_i}$ in terms of **ct** and **r**

$$: \frac{\partial}{\partial r_i} = \frac{\partial}{\partial r} = \frac{\partial ct}{\partial r}\frac{\partial}{\partial ct} + \frac{\partial r}{\partial r}\frac{\partial}{\partial r}$$

Taking the partial derivative of **ct** with respect to $\mathbf{r_i}$:

$$\frac{\partial ct}{\partial r_i} = \frac{\partial ct}{\partial r} = \frac{\partial}{\partial r}\left(v - r - r_s \ln\left(1 - \frac{r}{r_s}\right)\right)$$
$$\Rightarrow \frac{\partial ct}{\partial r_i} = -1 - r_s\left(\frac{1}{\frac{1}{r - r_s}}\right)\left(\frac{1}{r_s}\right)$$
$$\Rightarrow \frac{\partial ct}{\partial r_i} = -\frac{r}{r - r_s}$$

So, the $\mathbf{r_i}$ basis vector is:

$$\frac{\partial}{\partial r_i} = -\frac{r}{r - r_s}\frac{\partial}{\partial ct} + 1\frac{\partial}{\partial r}$$

Constructing the Schwarzschild metric in our new basis simply requires computing the dot products shown below:

$$\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v} = \left(\frac{\partial}{\partial ct}\right)\cdot\left(\frac{\partial}{\partial ct}\right)$$
$$\Rightarrow g_{vv} = \left(1 - \frac{r_s}{r}\right)$$

$$\frac{\partial}{\partial r_i}\cdot\frac{\partial}{\partial r_i} = \left(-\frac{r}{r - r_s}\frac{\partial}{\partial ct} + \frac{\partial}{\partial r}\right)\cdot\left(-\frac{r}{r - r_s}\frac{\partial}{\partial ct} + \frac{\partial}{\partial r}\right)$$
$$\Rightarrow \left(\frac{r}{r - r_s}\right)^2 g_{tt} + g_{rr} \Rightarrow \left(\frac{r}{r - r_s}\right)^2\left(1 - \frac{r_s}{r}\right) - \left(1 - \frac{r_s}{r}\right)^{-1}$$
$$\Rightarrow \left(\frac{r}{r - r_s}\right) - \left(\frac{r}{r - r_s}\right) \Rightarrow g_{r_i r_i} = 0$$

$$\frac{\partial}{\partial r_i}\cdot\frac{\partial}{\partial v} = \left(-\frac{r}{r - r_s}\frac{\partial}{\partial ct} + \frac{\partial}{\partial r}\right)\cdot\left(\frac{\partial}{\partial ct}\right)$$
$$\Rightarrow \left(-\frac{r}{r - r_s}\right) g_{tt} \Rightarrow \left(-\frac{r}{r - r_s}\right)\left(1 - \frac{r_s}{r}\right)$$
$$\Rightarrow \left(-\frac{r}{r - r_s}\right)\left(\frac{r - r_s}{r}\right) \Rightarrow g_{r_i v} = g_{v r_i} = -1$$

Plugging the above dot products into the metric, we arrive at the **in-going** Eddington-Finklestein metric:

$$\begin{bmatrix} \left(1 - \frac{r_s}{r}\right) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix} \tag{17}$$

For the sake of convenience and consistency in notation, we may also express this metric in the equivalent form:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) \mathrm{d}v^2 - (\mathrm{d}v\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}v) + r^2 d\Omega^2 \tag{18}$$

wherein the $r^2 d\Omega^2$ term represents the unchanged θ and ϕ coordinates of the spherically symmetric metric.

3.1.2 Outgoing EF Metric

The exact process described above can be applied to the outgoing coordinates **u** and $\mathbf{r_o}$, which define the **outgoing** EF metric as it is outgoing light-like geodesics that travel along constant coordinate curves, rendering the metric:

$$\begin{bmatrix} \left(1 - \frac{r_s}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix} \tag{19}$$

Or:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) \mathrm{d}v^2 + (\mathrm{d}v\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}v) + r^2 d\Omega^2 \tag{20}$$

Although these two metrics allow us to express both in-going and outgoing light beams, the two still appear to be disconnected at the Schwarzschild radius. That is, depending on our choice of metric, we either allow for the existence of in-going or outgoing light beams, but not both. This motivates the Kruskal-Szekeres (KS) coordinates.

3.2 Kruskal Szekeres Coordinates

In an attempt to unify these two metrics and allow our metric to span the entirety of spacetime, a good ansatz is to simply define our initial **ct** and **r** coordinates in terms of some combination of **v** and **u**. We may attempt this, starting from our definitions in 15 and 16:

$$v \equiv ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|$$

$$u \equiv ct - r - r_s \ln\left|\frac{r}{r_s} - 1\right|$$

Therefore, we may express **ct** as:

$$ct \equiv \frac{v + u}{2}$$

To express an r coordinate as a linear combination of **v** and **u**, we simply define r* as:

$$\frac{v - u}{2} = r + r_s \ln\left|\frac{r}{r_s} - 1\right| \equiv r^*$$

We follow a familiar process of expressing basis vectors as partial derivatives expanded with chain rule;

$$\frac{\partial}{\partial v} = \frac{\partial ct}{\partial v}\frac{\partial}{\partial ct} + \frac{\partial r}{\partial v}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{\partial r^*}{\partial v}\frac{\partial r}{\partial r^*}\frac{\partial}{\partial r} \Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{1}{2}\left(\frac{\partial r^*}{\partial r}\right)^{-1}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{1}{2}\left(\frac{r}{r - r_s}\right)^{-1}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{\partial}{\partial v} = \frac{1}{2}\frac{\partial}{\partial ct} + \frac{1}{2}\left(\frac{r - r_s}{r}\right)\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial u} = \frac{\partial ct}{\partial u}\frac{\partial}{\partial ct} + \frac{\partial u}{\partial v}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{\partial r^*}{\partial v}\frac{\partial r}{\partial r^*}\frac{\partial}{\partial r} \Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{1}{2}\left(\frac{\partial r^*}{\partial r}\right)^{-1}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial ct} + \frac{1}{2}\left(\frac{r}{r - r_s}\right)^{-1}\frac{\partial}{\partial r}$$

$$\Rightarrow \frac{\partial}{\partial u} = \frac{1}{2}\frac{\partial}{\partial ct} - \frac{1}{2}\left(\frac{r - r_s}{r}\right)\frac{\partial}{\partial r}$$

taking the necessary dot products;

$$\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial v}=\left(\frac{1}{2}\frac{\partial}{\partial ct}+\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)\cdot\left(\frac{1}{2}\frac{\partial}{\partial ct}+\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)$$
$$\Rightarrow \frac{1}{4}g_{tt}+\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2 g_{rr}\Rightarrow \frac{1}{4}\left(\frac{r-r_s}{r}\right)-\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2\left(\frac{r-r_s}{r}\right)^{-1}$$
$$\Rightarrow g_{vv}=0$$

$$\frac{\partial}{\partial u}\cdot\frac{\partial}{\partial u}=\left(\frac{1}{2}\frac{\partial}{\partial ct}-\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)\cdot\left(\frac{1}{2}\frac{\partial}{\partial ct}-\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)$$
$$\Rightarrow \frac{1}{4}g_{tt}+\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2 g_{rr}\Rightarrow \frac{1}{4}\left(\frac{r-r_s}{r}\right)-\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2\left(\frac{r-r_s}{r}\right)^{-1}$$
$$\Rightarrow g_{uu}=0$$

$$\frac{\partial}{\partial v}\cdot\frac{\partial}{\partial u}=\left(\frac{1}{2}\frac{\partial}{\partial ct}+\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)\cdot\left(\frac{1}{2}\frac{\partial}{\partial ct}-\frac{1}{2}\left(\frac{r-r_s}{r}\right)\frac{\partial}{\partial r}\right)$$
$$\Rightarrow \frac{1}{4}g_{tt}-\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2 g_{rr}\Rightarrow \frac{1}{4}\left(\frac{r-r_s}{r}\right)+\frac{1}{4}\left(\frac{r-r_s}{r}\right)^2\left(\frac{r-r_s}{r}\right)^{-1}$$
$$\Rightarrow g_{vu}=g_{uv}=\frac{1}{2}\left(\frac{r-r_s}{r}\right)=\frac{1}{2}\left(1-\frac{r_s}{r}\right)$$

and expressing the metric:

$$ds^2=\frac{1}{2}\left(1-\frac{r_s}{r}\right)(\mathsf{d}v\,\mathsf{d}r+\mathsf{d}r\,\mathsf{d}v)+r^2d\Omega^2$$

However, this metric appears to have a similar issue to our initial Schwarzschild metric: at $r=r_s$, the metric is undefined! We find that this issue arises from the presence of the logarithm in our expression for **r**:

$$\frac{v-u}{2}=r+r_s\ln\left|\frac{r}{r_s}-1\right|$$

Just like any other logarithmic expression, we can exponentiate both sides to avoid dealing with the coordinate singularity above:

$$\frac{v-u}{2r_s}=\frac{r}{r_s}+\ln\left|\frac{r}{r_s}-1\right|\Rightarrow \exp\left(\frac{v-u}{2r_s}\right)=\exp\left(\frac{r}{r_s}\right)\left|\frac{r}{r_s}-1\right|$$
$$\Rightarrow \exp\left(\frac{v}{2r_s}\right)\exp\left(\frac{-u}{2r_s}\right)=\exp\left(\frac{r}{r_s}\right)\left|\frac{r}{r_s}-1\right|$$

Now that we have rid our equation of logarithms, we may more clearly define our new coordinates. Rather than using v and u, we define V and U such that:

$$\begin{aligned} V &\equiv \text{exp}\left(\frac{v}{2r_s}\right) \\ U &\equiv -\text{exp}\left(\frac{-u}{2r_s}\right) \\ -VU &\equiv \text{exp}\left(\frac{r}{r_s}\right)\left|\frac{r}{r_s}-1\right| \end{aligned} \tag{21}$$

We define U with a negative sign in front of the exponential for the sake of convention, and simplicity in our future equations. If the reader is so inclined, they may omit the negative sign presently, only to find an extra negative in the upcoming derivation. At this point, given the number of previous coordinate transforms, every step of the following derivation may not be explicit. The approach outlined will be identical to those of the prior substitutions; if at any point, a step is unintuitive, it will be explained in depth. Explicitly, our steps are as follows: express our new basis as partial derivatives expanded with chain rule, take the necessary dot products, and express the metric in terms of the new coordinates.

$$g_{VV} = \frac{\partial}{\partial V}\cdot\frac{\partial}{\partial V} \Rightarrow \left(\frac{\partial v}{\partial V}\right)^2 g_{vv} => g_{VV} = 0$$

$$g_{UU} = \frac{\partial}{\partial U}\cdot\frac{\partial}{\partial U} \Rightarrow \left(\frac{\partial u}{\partial U}\right)^2 g_{uu} => g_{UU} = 0$$

$$\begin{aligned} g_{VU} = g_{UV} = \left(\frac{\partial v}{\partial V}\frac{\partial}{\partial v}\cdot\frac{\partial u}{\partial U}\frac{\partial}{\partial u}\right) &\Rightarrow \frac{\partial v}{\partial V}\frac{\partial u}{\partial U}g_{vu} \\ &\Rightarrow \frac{1}{2}\left(\frac{2r_s}{V}\right)\left(\frac{-2r_s}{U}\right)g_{vu} \\ \Rightarrow \left(\frac{-2{r_s}^2}{VU}\right)\left(\frac{r-r_s}{r}\right) &\Rightarrow \left(\frac{2{r_s}^2}{\text{exp}\left(\frac{r}{r_s}\right)\left|\frac{r-r_s}{r_s}\right|}\right)\left(\frac{r-r_s}{r}\right) \\ \Rightarrow \left(\frac{2{r_s}^2}{\text{exp}\left(\frac{r}{r_s}\right)}\right)\left(\frac{r_s}{r}\right) &\Rightarrow \frac{2{r_s}^3}{r\,\text{exp}\left(\frac{r}{r_s}\right)} \\ \Rightarrow g_{VU} = g_{UV} &= \frac{2{r_s}^3}{r}\text{e}^{\frac{-r}{r_s}} \end{aligned}$$

With these new coordinates, our completed metric can be expressed as:

$$\begin{bmatrix} 0 & \left(\frac{2{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right) & 0 & 0 \\ \left(\frac{2{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right) & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \tag{22}$$

As we can see, this new metric avoids the coordinate singularity at $r = r_s$; at this point, the terms g_{UV} and g_{VU} are nonzero and clearly defined as $\frac{2{r_s}^2}{\mathrm{e}}$. Our final required coordinate transform is for the sake of visualization on a spacetime diagram. In its current form, our basis vectors **V** and **U** are light-like vectors, given by the fact that their squared lengths are 0. Recall that this also means that they are at 45 degree angles with our hypothetical space-like coordinates. Therefore, in order to convert to space-like coordinates, we follow similar logic to that of our previous transforms between (**ct**, **r***) and (**v**, **u**).

We begin by defining our new time-like and space-like bases vectors T and X with the following transforms:

$$T \equiv \frac{V+U}{2} \longleftrightarrow V \equiv T + X$$
$$X \equiv \frac{V-U}{2} \longleftrightarrow U \equiv T - X$$

Following the usual steps for expressing the metric in a new basis:

$$\frac{\partial}{\partial T}\cdot\frac{\partial}{\partial T} = 2\left(\frac{\partial V}{\partial T}\frac{\partial U}{\partial T}\right)g_{UV} \Rightarrow 2(1)(1)\left(\frac{2{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right)$$
$$\Rightarrow g_{TT} = \frac{4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}$$

$$\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial X} = 2\left(\frac{\partial V}{\partial X}\frac{\partial U}{\partial X}\right)g_{UV} \Rightarrow 2(1)(-1)\left(\frac{2{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right) \Rightarrow g_{XX} = -\frac{4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}$$

Although the above transforms yield us an acceptable metric, for the sake of efficiency in visualization, we find it would be far more convenient to have transforms directly from the Schwarzschild coordinates **ct** and **r** to the time-like and space-like KS coordinates **T** and **X** independent from the various different light-like coordinates. This way, we avoid unnecessarily performing all of the transforms above when converting between Schwarzschild and KS coordinates. Luckily enough, given the expressions for the transforms we have already performed, this step is not too tedious. We begin by listing the transforms below:

$$T \equiv \frac{V+U}{2} \quad | \quad X \equiv \frac{V-U}{2}$$
$$V \equiv \exp\left(\frac{v}{2r_s}\right) \quad | \quad U \equiv -\exp\left(\frac{-u}{2r_s}\right)$$
$$v \equiv ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right| \quad | \quad u \equiv ct - r - r_s \ln\left|\frac{r}{r_s} - 1\right|$$

Expressing **T**:

$$T \equiv \frac{V+U}{2} \Rightarrow \frac{1}{2}\left(\exp\left(\frac{v}{2r_s}\right) - \exp\left(\frac{-u}{2r_s}\right)\right)$$
$$\Rightarrow \frac{1}{2}\left(\exp\left(\frac{ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|}{2r_s}\right) - \exp\left(\frac{-ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|}{2r_s}\right)\right)$$
$$\Rightarrow \frac{1}{2}\exp\left(\frac{ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\exp\left(\ln\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}\right) - \frac{1}{2}\exp\left(\frac{-ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\exp\left(\ln\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}\right)$$
$$\Rightarrow \frac{1}{2}\exp\left(\frac{ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}} - \frac{1}{2}\exp\left(\frac{-ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}$$

The absolute value symbols in our logarithms result in two independent $\pm$ signs, so our final expression is:

$$\mathbf{T} \equiv \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}}\left[\pm\mathrm{e}^{ct/2r_s} - \pm\mathrm{e}^{-ct/2r_s}\right]$$

We follow essentially the same process to render our final transform for **X**:

$$X \equiv \frac{V-U}{2} \Rightarrow \frac{1}{2}\left(\exp\left(\frac{v}{2r_s}\right) + \exp\left(\frac{-u}{2r_s}\right)\right)$$
$$\Rightarrow \frac{1}{2}\left(\exp\left(\frac{ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|}{2r_s}\right) + \exp\left(\frac{-ct + r + r_s \ln\left|\frac{r}{r_s} - 1\right|}{2r_s}\right)\right)$$
$$\Rightarrow \frac{1}{2}\exp\left(\frac{ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\exp\left(\ln\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}\right) + \frac{1}{2}\exp\left(\frac{-ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\exp\left(\ln\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}\right)$$
$$\Rightarrow \frac{1}{2}\exp\left(\frac{ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}} + \frac{1}{2}\exp\left(\frac{-ct}{2r_s}\right)\exp\left(\frac{r}{2r_s}\right)\left|\frac{r}{r_s} - 1\right|^{\frac{1}{2}}$$
$$\Rightarrow \mathbf{X} \equiv \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}}\left[\pm\mathrm{e}^{ct/2r_s} \pm \mathrm{e}^{-ct/2r_s}\right]$$

In summary, the Schwarzschild metric in KS coordinates is given by:

$$\begin{bmatrix} \left(\frac{4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{-4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix} \tag{23}$$

or equivalently as:

$$ds^2 = \left(\frac{4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right)\mathrm{d}T^2 - \left(\frac{4{r_s}^3}{r}\mathrm{e}^{\frac{-r}{r_s}}\right)\mathrm{d}X^2 - r^2 d\Omega^2 \tag{24}$$

The Kruskal Szekeres coordinates **T** and **X** are defined in terms of the Schwarzschild coordinates **ct** and **r** by the following transformation rules:

$$\begin{aligned} \mathbf{T} &\equiv \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s}-1\right)^{\frac{1}{2}}\left[\pm\mathrm{e}^{ct/2r_s} - \pm\mathrm{e}^{-\,ct/2r_s}\right] \\ \mathbf{X} &\equiv \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s}-1\right)^{\frac{1}{2}}\left[\pm\mathrm{e}^{ct/2r_s} \pm \mathrm{e}^{-\,ct/2r_s}\right] \end{aligned} \tag{25}$$

Note that these sums of exponentials c an also be expressed in terms of hyperbolic trigonometry:

$$\begin{gathered} \mathbf{T} \equiv \\ \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s}-1\right)^{\frac{1}{2}}\sinh{}^{ct}\!/_{2r_s} \\ \frac{1}{2}\mathrm{e}^{r/2r_s}\left(1-\frac{r}{r_s}\right)^{\frac{1}{2}}\cosh{}^{ct}\!/_{2r_s} \\ \mathbf{X} \equiv \\ \frac{1}{2}\mathrm{e}^{r/2r_s}\left(\frac{r}{r_s}-1\right)^{\frac{1}{2}}\cosh{}^{ct}\!/_{2r_s} \\ \frac{1}{2}\mathrm{e}^{r/2r_s}\left(1-\frac{r}{r_s}\right)^{\frac{1}{2}}\sinh{}^{ct}\!/_{2r_s} \end{gathered} \tag{26}$$

These coordinate lines give rise to the following **Kruskal Szekeres diagrams**:

The above diagrams explicitly show the quadripartite KS model of the Schwarzschild metric, with the latter simply using hyperbolic tangent function coordinate transforms to map infinities to the finite locations on the diagram. The “sections” of these diagrams are defined as follows: Section II represents the interior of a black hole (as time moves forwards, all future light-cones past the event horizon point to the singularity), Section IV represents the interior of a hypothetical “white hole” (as time moves forwards, all future light-cones past the event horizon seem to point out of the singularity, implying the opposite of a black hole), and Sections I and III represent parallel regions of exterior space-time, which extend to $r \to \infty$ and $r \to -\infty$ respectively. This implies that Sections I and III are infinitely far apart from one another, so the point at which all four regions collide must be a junction, or bridge of some sort. As this point is clearly defined, and its coordinate values are real, it must be considered a feasible object in the mathematics of General Relativity, and is monikered an **Einstein-Rosen Bridge** or, colloquially, a wormhole.

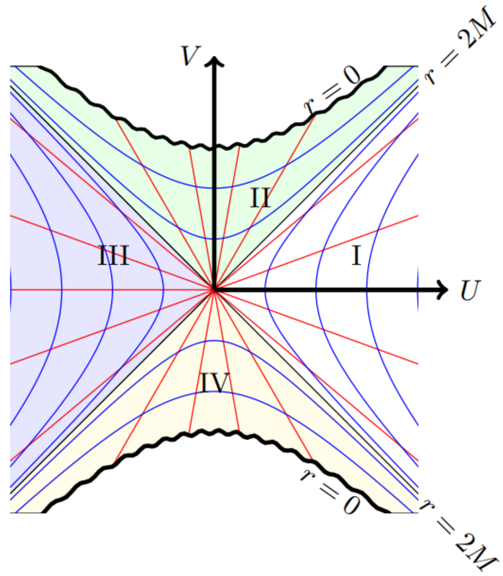

Figure 1: The Kruskal Szekeres Coordinate Diagram [1]

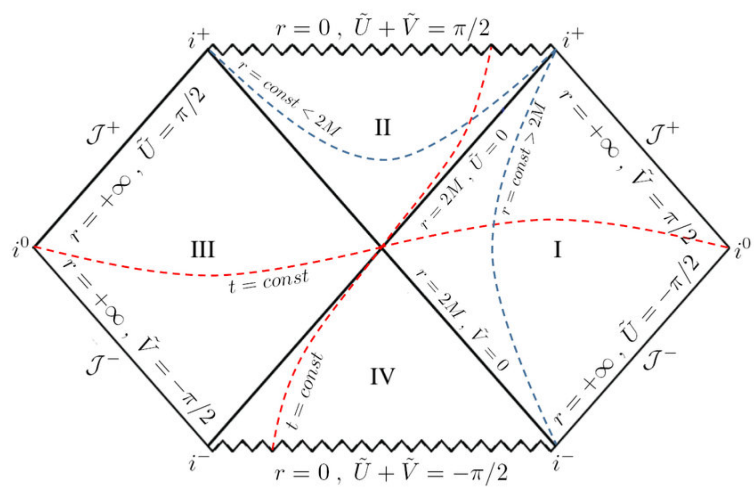

Figure 2: Penrose diagram for the Kruskal extension of Schwarzschild spacetime. [2]

4 Understanding Quantum Entanglement

We begin with a brief introduction to quantum entanglement. Quantum entanglement can best be described as the essence of quantum mechanics, the aspect of the field that creates its foundations in the first place. Seeing as this article is primarily concerned with developing the rigorous mathematical understanding of the concepts discussed, we will begin with the basic formalism and work our way up to a more developed mathematical language.

4.1 Formalism and Dirac Notation

Mathematical understanding of quantum entanglement necessitates a basic understanding of linear algebra, operators, and matrices in Dirac Notation. Quantum properties are represented as state vectors $|state\rangle$. Operators are represented by matrices: $M|\alpha\rangle = \lambda|\alpha\rangle$ where α represents an eigenvector of the M operator and λ represents the eigenvalue.

For some diagonal operator M acting on some state vector $|\alpha\rangle$, where $|\alpha\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we may use the rules of matrix multiplication rooted in linear algebra:

$$\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} \\ 0 \end{pmatrix} = M_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Above, M_{11} is an example of the eigenvalue of the M operator acting on the $|\alpha\rangle$ vector.

4.2 The Spin Operator

For the sake of simplicity, we will start with the example of spin. We begin with an electron with spin state $|u\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We then define the spin operators:

$$\sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The above operators can be applied to any arbitrary state vector $\hat{n}$. The spin state of $\hat{n}$ is given by $\sigma \cdot \hat{n}$:

$$\begin{aligned} \sigma \cdot \hat{n} &= \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (n_3) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (n_2) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (n_1) \\ &= \begin{pmatrix} n_3 & 0 \\ 0 & -n_3 \end{pmatrix} + \begin{pmatrix} 0 & -i(n_2) \\ i(n_2) & 0 \end{pmatrix} + \begin{pmatrix} 0 & n_1 \\ n_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} n_3 & n_1 - i(n_2) \\ n_1 + i(n_2) & -n_3 \end{pmatrix} \end{aligned}$$

For any n, $(\sigma \cdot \hat{n})^2 = 1$, given by the fact that probability that a particle has spin in any direction $\hat{n}$ has to be equal to 1. We may write this probability equation in Dirac Notation:

$$\rho = |\langle n| \hat{\sigma} |n\rangle|^2 = 1$$

The σ operator measures a spin in a given direction, with the different indices corresponding to different coordinates in the Cartesian plane. We may use elementary linear algebra to construct a table:

Table of the σ operator acting on different electron states
$\sigma_1 \lvert u\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lvert d\rangle$
$\sigma_1 \lvert d\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lvert u\rangle$
$\sigma_2 \lvert u\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i \lvert d\rangle$
$\sigma_2 \lvert d\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i \lvert u\rangle$
$\sigma_3 \lvert u\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lvert u\rangle$
$\sigma_3 \lvert d\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \lvert d\rangle$

4.3 The Dual Electron System

Thus far we have been working under the assumption of a single-electron system, whose spin is measured as a superposition of all possible states for that single particle.

We now consider some system with two electrons. Seeing as we have a set of operations to be done on the spin states of electron 1, we must have an analogous set for electron 2. We may refer to these analogous operators as the τ operators, and define their action on spin states of electron 2 as identical to the action of the σ operators on electron 1. With a two electron system, we have not two but four distinct basis states, each with their own complex probability amplitudes: $\lvert uu\rangle, \lvert ud\rangle, \lvert du\rangle, \lvert dd\rangle$.

An entangled state is, in essence, a relationship between the two spin states of the electrons themselves which has the property that knowledge about electron 1's state informs the observer of electron 2's state.

One such state is known as the singlet state, and it is reflected mathematically in terms of our new basis vectors as:

$$\frac{\lvert ud\rangle - \lvert du\rangle}{\sqrt{2}} \tag{27}$$

where the $\frac{1}{\sqrt{2}}$ factor exists for the purpose of normalization (that is, the squares of probability amplitudes must add up to 1 - the total possible probability).

The singlet state represented above is an example of a maximally entangled state, best defined conceptually as a state in which learning all the information about one partner effectively gives the observer all the information about the other partner, without having to do any direct measurement on the partner state individually.

4.4 Entropy in Statistical and Quantum Mechanics

The density matrix can be defined as the mathematical formulation for some system comprised of some subsystems with distinct probabilities. Computing the density matrix for a system is essentially the quantum analog for defining the probability density function of a system. The concept of entropy plays an important role in understanding a system. Entropy is simply a measure of the amount of disorder in a system, or, in other words, the amount that we do not know about a system. Although its basis in classical physics arises from thermodynamics, conceptually, entropy is ever-present in all fields of physics. Entropy plays a notable role in understanding quantum entanglement, serving as a measure of the degree of entanglement in a system. To further understand it, we may carry out a rather simple derivation for an entropy equation in a classical system and generalize to quantum mechanics.

We begin with the most fundamental idea that entropy is a direct measure of the amount of information unavailable to the observer. Therefore, it must be proportional to the number of bits of information in a system. For a classical, binary system with two possible states and **m** terms, the net number of states is given by:

$$P = 2^m$$
$$S = \ln(P) = m \ln(2)$$

To generalize to systems statistically significant state probabilities, we simply take the variable probabilities of basis states into account. Rather than binary bits with the 100% probability of being in two possible states, we allow entropy to be weighted based on the probability of said states given by some probability density function P_m. In order for more probable states to be weighted more, our new entropy can be defined as:

$$S = P_i \ln(P_i)$$

with an implicit summation over i for all possible states. Therefore, for systems in which the observer knows the state of the system with complete certainty, the entropy for the system simplifies to:

$$S = P_i \ln(P_i)$$
$$\Rightarrow S = 1 \ln(1)$$

Conceptually, this would translate to an observer with total information, which is consistent with our example. To generalize to quantum mechanics itself, we must essentially convert to the notation of quantum mechanics. Asserting that probability density is an observable, we may use the standard assumption that all observable quantities are eigenvalues of hermitian operators in quantum mechanics to build an operator for probability density known as the **density matrix**. Recall that the probability density function for some arbitrary state $|\Psi\rangle$: is given by the Gibbs entropy:

$$\rho(x) = |\psi(x)|^2 = |\psi(x)\psi^*(x)| = |\psi\rangle\langle\psi|$$

To represent all probability space we may use a matrix containing a probability amplitude for each distinct state. Now, rather than a summation over some arbitrary index i, our new equation for entropy simply takes the trace of the hermitian matrix ρ, and we are left with the von Neumann formula for the entropy of quantum mechanical systems:

$$S = -\,\mathrm{Tr}(\rho - \ln \rho) \tag{28}$$

5 AdS/CFT Explained

The connections between general relativity, geodesics, and quantum entanglement rely on the idea of the AdS/CFT correspondence. This postulate has been at the forefront of physics for the last few decades, and for good reason. It allows us to refine and develop our understanding of the connections and potential unification of quantum mechanics and gravity.

We may begin our discussion of the AdS/CFT correspondence with a brief description of the terms themselves. It is a sort of duality between the General Relativity of Anti-de Sitter space (spacetime with a negative curvature), and a scale-invariant Quantum Field Theory at the boundary of this spacetime.

AdS/CFT states that, at large *N* limits of specific conformal field theories in *d* dimensions can be used to holographically describe theories of quantum gravity in a *d+1* dimensional *AdS* space.

At its core, the AdS/CFT correspondence essentially allows us to explicitly mathematically test our theories of quantum gravity, and come up with the implications that they have on quantum mechanical theories. This rather abstract concept can be made much clearer in the following image:

6 RT Formulate and ER=EPR

6.1 Ryu Takayanagi Formulate

It was Ryu and Takayanagi that postulated the mathematical interrelation between entanglement entropy in *d+1* dimensional scale-invariant QFTs (CFTs) and Bekenstein-Hawking entropy of black holes in *d+2* dimensional Anti-de Sitter space [4]. RT begin by expressing the analog for the von-Neumann entanglement entropy (28) in relativistic quantum field theory found in [5, 4]:

$$S = -\,\mathrm{Tr}\,\rho \ln \rho = \frac{\partial}{\partial n}\,\mathrm{Tr}\,\rho^n \bigg|_{n=1} = -\frac{\partial}{\partial n}\left(\xi^{-2}a^2\right)^{\frac{1}{24}\left(n-\frac{1}{n}\right)} \bigg|_{n=1} = -\frac{1}{12}\ln \xi^{-2}a^2 = \frac{c}{6}\cdot \mathcal{A} \cdot \ln\frac{\xi}{a} \tag{29}$$

Where $\mathcal{A}$ represents the number of boundary points of the field theory and c represents the central charge. RT then build an area-entropy relation law shown in (30) out of the infamous Bekenstein-Hawking Entropy shown in [6], and compare their expected computations for the area of a minimal surface in AdS_{d+2} to the values for the entropy of a quantum field theory in $d+1$ predicted by (29).

$$S_A = \frac{\text{Area of } \gamma_a}{4{G_N}^{(d+2)}} \tag{30}$$

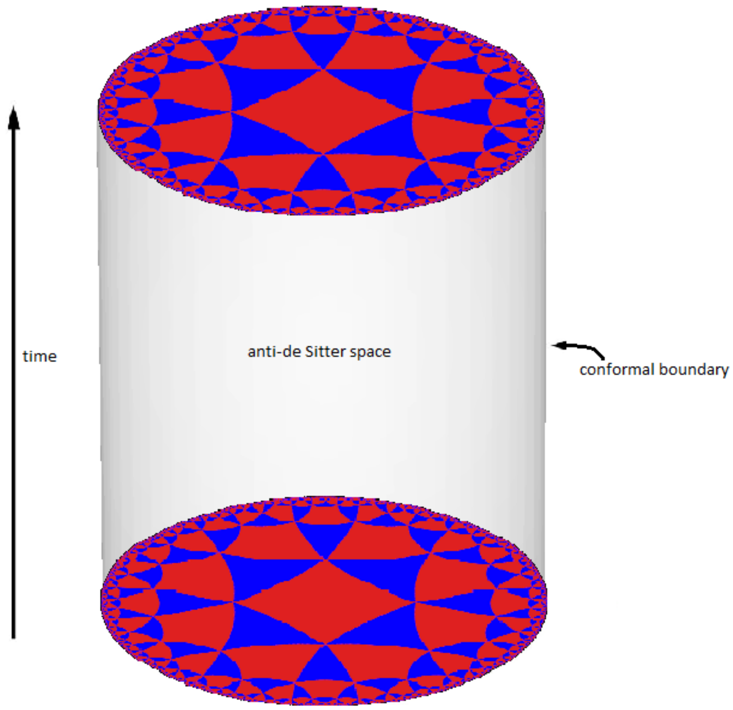

Figure 3: A visualization of (2+1)-dimensional anti-de Sitter space. Based on a similar picture in [3]. In the above image, we can imagine testing theories of quantum gravity on the interior volume of the cylinder. Theories of entanglement exist on the "conformal boundary" shown above, and the relationship between the two resides in the relationships between the coupling (or, range) of each.

wherein γ_a is defined as the static, *d* dimensional minimal surface that serves as a holographic screen in AdS$_{d+2}$ and $4{G_N}^{(d+2)}$ is Newton's gravitational constant in *d+2* dimensions [4]. RT found that their 'area relation' connecting entanglement to gravity held true to a promising degree of accuracy, expanding their work to the entanglement entropy of $\mathcal{N} = 4$ super Yang-Mills theory with finite Temperature in terms of the area (or length) of a minimal boundary surface in AdS$_{d+2}$. The more rigorous mathematical expression of these ideas by Shinsei Ryu and Tadashi Takayanagi can be found in [4].

6.2 ER=EPR

ER=EPR allows us to understand the relationships between two entangled black holes and wormholes. It has been extensively confirmed that, in the mathematical model of AdS, the existence of entanglement between two black holes implies the existence of a wormhole between them. The RT formulate can be used to extend this case from the theoretical AdS space to more general spacetime models. However, when two black holes are created independent of each other, no entanglement and thus no wormhole exists. We present the heuristic argument for the validity of ER=EPR under the assumption of the RT formulate found in [7]. This can be simply done assuming two systems separated by a RT surface (or, minimal boundary) between them. In this

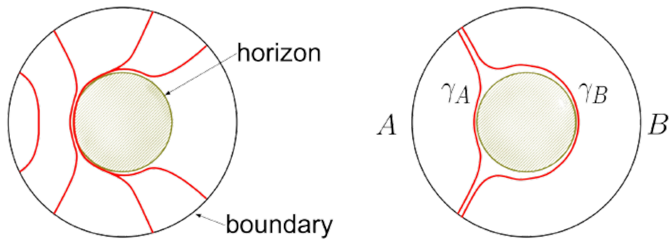

Figure 4: An illustration depicting different minimal surfaces γ_a and γ_b wrapping different portions of the horizon from [4]

situation, given the RT formulate, it follows that the entanglement entropy between the two systems themselves could be easily accounted for by the area of the minimal RT surface between them. If, however, one were to place entangled objects on either side of the boundary, the area of our previous RT boundary would not longer be sufficient to calculate the overall entanglement entropy of the system. Therefore, it becomes clear that, assuming the validity of the RT formulate, it is simply the area of the minimal surface that must change. We may visualize this change as a "stretch" in the RT boundary.

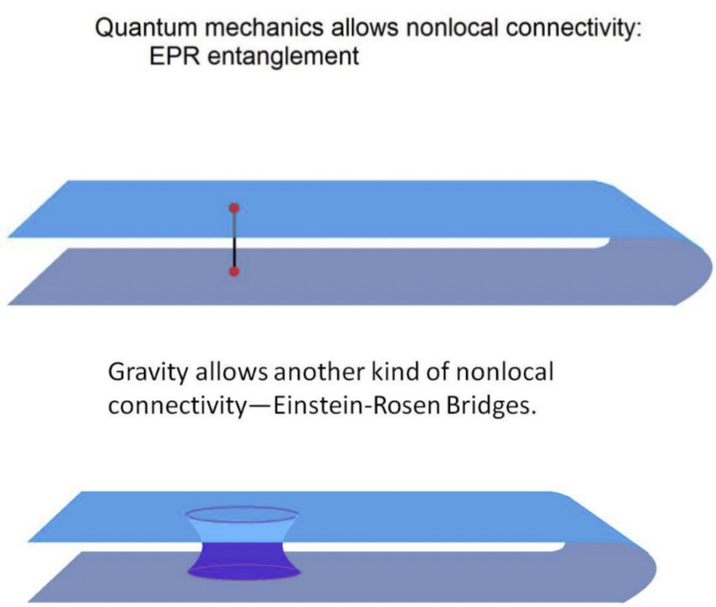

Figure 5: An illustration of the central idea behind ER=EPR [7]

Now, we replace the "entangled objects" in our example with a pair of black holes. If no entanglement exists between the two, it is evident that the minimal RT surface between is identical to that of the un-entangled objects. However, if there exists an ERB between the two black holes, there must exist some form of entanglement, and the additional "stretch" of the minimal RT surface may be visualized as "wrapping" the wormhole itself.

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