# PROOF OF A FUNCTIONAL EQUATION BY USING GRAPHS AND SET THEORY

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**Abstract**

There are different ways to approach in solving functional equation problems in mathematics. In this paper, Set theory and graph theory techniques, which are rarely encountered in the solutions of typical function problems, are used to construct our proof. At the same time, our approach differs from those commonly used problems solved by graph theory, as the solution is reached by examining uncountable graphs with the help of set theory.

**Keywords: cardinal, ordinal, graph theory, set theory, functions**

## 1.Introduction

The motivating problem for this work is the following task:

Prove that there exists a function $f : \mathbb{R}^2 \to \mathbb{Z}$ such that for every $x, y \in \mathbb{R}^2$ with $x \neq y$, then $f(x) = f(y) \Rightarrow d(x, y) \notin \mathbb{Q}$.

In more informal terms, the function $f$ is going to assign colors[1] to all points in the real $2$ dimensional space, and we must ensure that any 2 points that have the same color do not have a rational distance between them.

Unlike typical function problems in which the answer is shown by example, our proof will be grounded in a combination of graph theory and set theory. We will utilize a theorem proven by Erdős and Hajnal(1966). In Section , we provide some important definitions and improved explanations of lemmas and claims. In Section , we present the main proof of the theorem ,using the definitions,claims and lemmas we provided in Section 2. Then, in Section , we apply this theorem to the present task and discuss why it plays such a crucial role.

## 2. Definitions and Lemmas

We are going to give some definitions and prove related lemmas that are necessary for proof. We denote a graph $\mathscr{G} = \langle g, G\rangle$ where $g$ is the set of vertices of the graph and $G$ is the set of edges, each represented as a set of exactly 2 vertices.

**Definition 1:** $Chr(\mathscr{G})$ is the least cardinal $\alpha$ such that $g$ is the union of $\alpha$ sets, where no two elements of same set are connected by an edge in $G$.

$Col(\mathscr{G})$ is the least cardinal $\beta$ such that $g$ has a well ordering $<$ satisfying the condition that for every vertex $x \in g$, the number of those vertices $y < x$ and $\{x, y\} \in G$ is less than $\beta$.

**Lemma 1:** For every simple graph $\mathscr{G}$, $Chr(\mathscr{G}) \leq Col(\mathscr{G})$.

**Proof:** For every vertex $v \in g$ let us define a function[2] $f_1 : G \to \beta$ by transfinite induction on $v$ as: Assuming that $f_1(x)$ is defined for every $x \in g$ and $x < y$

[1]There may be a countably infinite number of colors.

[2]We add subscripts 1-6 to the function symbols to indicate distinctions among the original function for the problem and other helper functions. There is no order intended from the values of the subscripts other than order of introduction in this work.

(where the relation $<$ denotes the well ordering of the set $g$, which was mentioned in the definition of coloring number), let $f_1(v) = y$ where $y$ is the smallest element of $\boldsymbol{\beta \setminus \bigcup\{f_1(x) : (x < v) \wedge \{x, v\} \in G\}}$. By smallest element, we mean the element satisfying the relation
$y \cap (\beta \setminus \bigcup\{f_1(x) : x < v \wedge \{x, v\} \in R_G\}) = \emptyset$.
For ease of notation define, $N_v$ to be $N_v = \{ a : a \in g \wedge \{a, v\} \in G \wedge a < v\}$.

Define $I_v$ to be image set of $N_v$ under the function $f_1$. More formally written, $I_v = \{s : s \in \beta, \exists z, (z \in N_v \wedge f_1(z) = s\}$. If we can prove that $\beta/I_v$ is not empty, then there exist a $y$ such that $(y \in \beta/I_v) \wedge (y \cap \beta/I_v) = \emptyset$(by Axiom of Foundation).

Suppose, for contradiction, that $\beta/I_v = \emptyset$ for some $v$.
So $\beta \subseteq I_v \Rightarrow |\beta| \leq |I_v|$. Also all elements of $I_v$ are an image of some element of $N_v$ under $f_1$, so we know that $f_1$ is a surjective function from $N_v$ to $I_v$. This means that $|I_v| \leq |N_v|$. From the definition of the coloring number of $\mathscr{G}$, we know that $|N_v| < |\beta|$, which leads to a contradiction. So, we can conclude that $\beta/I_v$ is not empty. So, such a $y$ exists.

Now, we show that this $y$ is unique. Suppose, for contradiction, that there exist another $x \in \beta/I_v \wedge (x \cap \beta/I_v) = \emptyset$. We know that $x \neq y$ and by trichotomy of ordinals, either $x \in y$ or $y \in x$. Without loss of generality, assume that $y \in x$. Also, $y \in \beta/I_v$ as well. From this we get $y \in (x \cap \beta/I_v) \neq \emptyset$, which leads to a contradiction. So, uniqueness of $y$ is proved as well.

Now, let us define sets $M_\alpha$ such that for all $\alpha \in \beta$,
$M_\alpha = \{v : v \in g \wedge f_1(v) = \alpha\}$. Since every vertex of the graph is attained to an element of $\beta$ under $f_1$, $\bigcup M_\alpha = \beta$. Also if $a \neq b$ and $f_1(a) = f_1(b)$ for any two vertices of the graph, then there is not any edge between $a$ and $b$ in the graph $\mathscr{G}$. Suppose not. Let us say there exist two vertices on the graph such that $a \neq b$ and $f_1(a) = f_1(b)$. Without loss of generality, assume that $a < b$ in the well ordering of $g$. So, $f_1(b) = y$ such that $y \in \beta/I_v$, which means $f_1(b) \notin I_v$. On the other hand, $a < b \wedge (a, b) \in G$, so $a \in N_v \Rightarrow f(a) \in I_v$. This contradicts the fact that $f_1(a) = f_1(b)$.

The partition of $g$ into the sets $M_\alpha$ such that $\alpha \in \beta$ satisfies the condition that every set of the partition does not contain any two elements that have an edge between them. This means the set $g$ is partitioned in to $|\beta|$ sets that satisfy the condition of chromatic number property meaning that
$Chr(\mathscr{G}) \leq Col(\beta)$. ■

**Note:** In the remaining partof the proof, we will denote the cardinality of the vertex set of the graph by alpha. Meaning that $|g| = \alpha$. Also we are going to deal with graphs that have the property that$|g| = \alpha \geq |\omega|$ ,($\alpha$ denotes set $g$'s cardinality) where $|\omega|(\aleph_0)$ is the cardinality of the natural numbers. Although this property will not be used in all of the remaining lemmas, it will be applied in some of them. For instance, in the proof of Lemma 3, we mentioned about an ordinal $\xi < \alpha$ and also mentioned $\xi + 1$ as well.($+$ sign is used to denote ordinal addition) But $\xi + 1 < \alpha$ satisfies for all $\xi < \alpha$ if only if $\alpha$ is an infinite cardinal. Here, we have indirectly utilized this property.

**Definition 2**: Let us define $v(x, g', \mathscr{G})$ to be the set of neighbors of $x$ located in a derived subgraph of $\mathscr{G}$ defined by the vertices $g'$. And let $|v(x, g', \mathscr{G})| = \gamma(x, g', \mathscr{G})$.A subset $g' \subseteq g$ is said to be $\tau$-*closed* in $\mathscr{G}$ (for a fixed cardinal number $\tau$) if $\gamma(x, g', \mathscr{G}) \geq \tau$ implies $x \in g'$, for any $x \in g$.

**Claim 1**: The intersection of any number of $\tau$-*closed* subsets is again a $\tau$-*closed* subset.

**Proof**: Suppose, for contradiction, that this is not the case. Let us call $S$ an intersection of some $\tau$-*closed* sets in $\mathscr{G}$ that is not $\tau$-*closed*. Then for $S$, there exists an element $x \in g$ such that $x \notin S$ and $\gamma(x, S, \mathscr{G}) \geq \tau$. Since $S$ is the intersection of some $\tau$-*closed* subsets, there exists a $\tau$-*closed* subset $M$ such that $x \notin M$ and $S \subseteq M$. On the other hand, $\gamma(x, S, \mathscr{G}) \geq \tau$ implies $\gamma(x, M, \mathscr{G}) \geq \tau$ as well. This contradicts the fact that $M$ is $\tau$-*closed*. ■

Considering the fact that $g$ itself is $\tau$-*closed* for any $\tau$ and also Claim 1, for every $g' \subseteq g$ there exists a minimal $\tau$-*closed* subset containing $g'$. This will be called $\tau$-*closure* of $g'$ in $\mathscr{G}$ and it will be denoted by Clos$(g', \mathscr{G}, \tau)$.
It is also worth noting that because $|g| = Card(g) = \alpha$, there exists a bijective function from elements of $g$ to $\alpha$. Call this function $f_2$. Now, we are going to well-order the elements of $g$ as follows: For the previously mentioned function $f_2 : g \to \alpha$, and for any two different $x, y \in g$ , $\boldsymbol{x < y \Leftrightarrow f_2(x) < f_2(y)}$.(Also we are going to use this function $f_2$ and its property in the proof of Lemma $2$ as well.)

**Definition 3**:We are going to define a sequence $g_\xi$, for $\xi < \alpha$ of subsets of $g$ by transfinite induction on $\xi$ as follows: Assume $g_\zeta$ is defined for every $\zeta < \xi$ for some $\xi < \alpha$. Put $h_\xi = \bigcup\limits_{\zeta<\xi} g_\zeta$. If $h_\xi = g$ put $g_\xi = \emptyset$. If $g/h_\xi \neq \emptyset$, let $x_\xi$ to be the least element of $g/h_\xi$ in the well ordering and put $g_\xi = Clos(h_\xi \cup \{x_\xi\}, \mathscr{G}, \tau)/h_\xi$.

**Lemma 2**: The sequence $g_\xi$, $\xi < \alpha$ is disjointed and $g = \bigcup\limits_{\xi<\alpha} g_\zeta$ .

**Proof**: First we are going prove that the sequence is disjointed. Suppose, for contradiction, that for any two different $\xi, \zeta < \alpha$ that $g_\xi$ and $g_\zeta$ have an element in common. WLOG assume that $\zeta < \xi$, from the definition we know that either $g_\xi = \emptyset$ or $g_\xi \cap h_\xi = \emptyset$. In both cases $g_\xi \cap h_\xi = \emptyset$. Since $\zeta < \xi$, $g_\zeta \subseteq h_\xi$. So $g_\xi \cap g_\zeta = \emptyset$, which leads to a contradiction. Therefore, the sequence must be disjointed.
For the second part, we are going to show that $f_2(x_\beta) \geq \beta$, where $f_2$ is the previously mentioned function from $g$ to $\alpha$ that was used to well order the set $g$. We are going to use transfinite induction on $\beta$ to prove this claim. The base case is trivial because the empty set is always a subset of any set. So we need to prove that if for any ordinal $\zeta < \beta$, $f_2(x_\zeta) \geq \zeta$ implies $f_2(x_\beta) \geq \beta$.
First we need to show that $x_\beta > x_\zeta$, for any $\zeta < \beta$. Suppose, for contradiction, that there exist a $\zeta$ such that $x_\beta \leq x_\zeta$. We know that $x_\beta \notin h_\beta$. This implies that $x_\beta \notin h_\epsilon$ for any $\epsilon < \beta$. Now, $x_\zeta$ needs to be the least element of $g$ that is not in $h_\zeta$. But we also know that $x_\beta \notin h_\zeta$ as well. So this leads to $x_\zeta \leq x_\beta$. From the contradiction supposition, we also know that $x_\zeta \geq x_\beta$. So $x_\zeta = x_\beta$. But this gives a contradiction from the disjointedness property of the sequence. We can conclude that $x_\beta > x_\zeta$, for any $\zeta < \beta$. Which also means $f_2(x_\beta) > f_2(x_\zeta)$, for any $\zeta < \beta$. Also from induction hypothesis $f_2(x_\beta) > f_2(x_\zeta) \geq \zeta$. So, $f_2(x_\beta)$ is an ordinal such that $\zeta < \beta$ implies $f_2(x_\beta) > \zeta$. This itself is sufficient to show that $f_2(x_\beta) \geq \beta$. Now take any element $v \in g$ and call $f_2(v) = \zeta$. Now, take a $\xi < \alpha$ such that $\zeta < \xi$. (This is possible since $\alpha$ is an infinite cardinal) $f_2(x_\xi) \geq \xi$ meaning that for any vertex of the graph $y$, if $f_2(y) < \xi$ then $y \in h_\xi$.

Since $f_2(v) = \zeta < \xi$ this means $v \in h_\xi$. This result suggests that every element of $g$ is also in $\bigcup g_\xi$ $(\xi < \alpha)$. The opposite is true as well. From this we conclude that $g = \bigcup g_\xi$ $(\xi < \alpha)$. ■

**Lemma 3**: Assume $x \in g_\xi$, $\xi < \alpha$. Then,
$\gamma(x, h_\xi, \mathscr{G}) \leq \tau$ if $\tau \geq |\omega|$,
$\gamma(x, h_\xi, \mathscr{G}) < \tau$ if $\tau < |\omega|$.

**Proof**: For every $\xi < \alpha$, $g_\xi = h_{\xi+1}/h_\xi$ as well.($+$ is used to denote ordinal addition).Since $\xi + 1 = \xi \cup \{\xi\}$. This means $h_{\xi+1} = h_\xi \cup g_\xi$. Also $h_{\xi+1}$ is always $\tau$-*closed*. Since $g_\xi = Clos(h_\xi \cup \{x_\xi\}, \mathscr{G}, \tau)/h_\xi$, $h_{\xi+1} = Clos(h_\xi \cup \{x_\xi\}, \mathscr{G}, \tau)$ and $Clos(h_\xi \cup \{x_\xi\}, \mathscr{G}, \tau)$ is $\tau$-*closed* by definition. Also $\gamma(x, h_{\xi+1}, \mathscr{G}) < \tau$ for every $\xi < \alpha$, $x \in g_\zeta$ provided that $\xi < \zeta$. That is because the sequence $g_\xi$ is disjointed. So, if $x \in g_\zeta$ such that $\xi < \zeta$, $x \notin h_{\xi+1} = h_\xi \cup g_\xi$ and also $h_{\xi+1}$ is always $\tau$-*closed*. So, $x \notin h_{\xi+1}$ implies $\gamma(x, h_{\xi+1}, \mathscr{G}) < \tau$. We want to show that $\gamma(x, h_\xi, \mathscr{G}) \leq \bigcup\limits_{\zeta<\xi} \gamma(x, h_{\zeta+1}, \mathscr{G})$. Equivalently, $v(x, h_\xi, \mathscr{G}) \leq \bigcup\limits_{\zeta<\xi} v(x, h_{\zeta+1}, \mathscr{G})$. For every $a \in h_\xi$ for some $\zeta < \xi$, $a \in g_\zeta$. Also $g_\zeta \subseteq h_{\zeta+1}$, so every element of $v(x, g_\zeta, \mathscr{G})$ is also an element of $v(x, h_{\zeta+1}, \mathscr{G})$. That is why $v(x, h_\xi, \mathscr{G}) \subseteq \bigcup\limits_{\zeta<\xi} v(x, h_{\zeta+1}, \mathscr{G})$. This leads to the desired result. Since $\gamma(x, h_{\xi+1}, \mathscr{G}) < \tau$ for every $\zeta < \xi$, the union of the increasing sequence of cardinals $< \tau$ is $\leq \tau$ if $\tau$ is infinite and is $< \tau$ if $\tau < |\omega|$. ■

**Claim 2:** Assume $H$ to be a set of sets. Suppose any $a \in H$ , $|a| \leq \kappa$ for a cardinal number $\kappa$ and $|H| \leq \alpha$ then $|\bigcup\limits_{a \in H} a| \leq |\kappa \times \alpha|$.

**Proof:** We are going to prove the desired result by saying that there exist a surjective function from a subset of $\kappa \times \alpha$ to $\bigcup\limits_{a \in H} a$. For all sets $a \in H$, we use the function $f_3$ ,which is a bijection from set $H$ to a cardinal $\beta \leq \alpha$,to denote every $a$ with a unique $f_3(a) \leq \alpha$. ( Existince of such function is true since $|H| \leq |\alpha|$).Again, we denote every element $x$ of $a$(an element of $H$) by a function $g_a$ ,which is a bijection from $a$ to a cardinal $\lambda \leq \kappa$ , as a unique $g_a(x)$ for an $g_a(x) \leq \kappa$ .At last, we denote each element of $\bigcup\limits_{a \in H} a$ as follows :
For every $x$ such that $x \in a$ for $a \in H$.We denote $x$ to be $(g_a(x), f_3(a))$ .This is a injective function from $|\bigcup\limits_{a \in H} a$ to $\kappa \times \alpha$. This is also equivalent to say that there exist a surjective function from a subset of $\kappa \times \alpha$ to $\bigcup\limits_{a \in H} a$. We can conclude that $|\bigcup\limits_{a \in H} a| \leq |\kappa \times \alpha|$. Also if any of $\kappa, \alpha$ is an infinite cardinal, $|\kappa \times \alpha| = \max(\kappa, \alpha)$. ■

**Definition 4:** For any cardinal $\kappa$ ,$\kappa^+$ is the least cardinal greater than $\kappa$.

**Lemma 4:** Assume $g' \subseteq g$ , $|g'| \leq \kappa$ , $|\omega| \leq \kappa$ and $\tau < |\omega|$. (We denote $g$ to be the vertex set of $\mathscr{G}$). If graph $\mathscr{G}$ doesn't contain a complete bipartite graph $[\kappa^+, \delta]$ for a $\delta < \tau$. Then $|Clos(g', \mathscr{G}, \tau)| \leq \kappa$.(By complete bipartite graph $[\gamma, \delta]$, we mean there exist two sets $A, B$ such that $|\gamma| = A$ and $|\delta| = B$ and for every $a \in A, b \in B$ there exist an edge between $a$ and $b(\{a, b\} \in G)$)

**Proof:** We define a sequence $A_\xi$ , for $\xi < \omega$ of subsets of $g$ by transfinite induction on $\xi$ as follows.
Assume $A_\zeta$ , is defined for every $\zeta < \xi$ for some $\xi < \omega$ .Put $B_\xi = g' \cup \bigcup_{\zeta<\xi} A_\zeta$ and
$A_\xi = \{x \in g \setminus B_\xi : \tau(x, B_\xi, \mathscr{G}) \geq \tau\}$.
We are going to prove by transfinite induction on $\xi$ that $|A_\xi| \leq \kappa$ for every $\xi \leq \omega$. Assume that the induction hypothesis is true for every $\zeta < \xi$ for some $\xi \leq \omega$. Suppose contrary that $|A_\xi| \geq \kappa^+$. Our first observation will be that $|B_\xi| \leq \kappa$. By using both the induction hypothesis and **Claim 2** , $|\bigcup_{\zeta<\xi} A_\zeta| \leq |\kappa \times \omega| = \kappa$. Also $|g'| \leq \kappa$ , so $B_\xi = g' \cup \bigcup_{\zeta<\xi} A_\zeta$ leads to $|B_\xi| \leq \kappa$.
Let's call $M_\xi = \{x | x \subseteq B_\xi, |x| = \tau\}$ , our first step is to prove that $|M_\xi| \leq \kappa$. If $M_\xi$ is a finite set , then the number subset of $M_\xi$ is also finite which is obviously $< \kappa$. Suppose $|M_\xi| = \alpha \geq |\omega|$ , it is known a fact that any infinite set's finite cartesian product's is equal to sets cardinality. Which means $|M_\xi| \leq |B_\xi^\tau| = |B_\xi| \leq \kappa$.( where $B_\xi^\tau$ means the $\tau$ times cartesian product of the set $B_\xi$). After this step, for every $x \in A_\xi$ , we choose a subset of $v(x, B_\xi, \mathscr{G})$, such that these subset $S_x$ satisfies the property that $S_x \subseteq v(x, B_\xi, \mathscr{G}), |S_x| = \tau$.(There exist such set because $v(x, B_\xi, \mathscr{G}) \geq \tau$ from the definition of $A_\xi$). Now, we define a function $f_4$ from $A_\xi$ to $M_\xi$ , such that for any $x \in A_\xi$, $f_4(x) = S_x$ . For every $y \in M_\xi$ , Let $f_4^{-1}(y) = \{x : x \in A_\xi, f_4(x) = y\}$. We are going to prove that there exist such $y \in M_\xi$ such that $|f_4^{-1}(y)| \geq \kappa^+$ . If not , then for all $y \in M_\xi$ $|f_4^{-1}(y)| \leq \kappa$. Under the function $f_4$ every element of $A_\xi$ has its image in $M_\xi$. So , $\bigcup_{y\in M_\xi} f_4^{-1}(y) = A_\xi$. Using **Claim 2** we can see that $|A_\xi| = |\bigcup_{y\in M_\xi} f_4^{-1}(y)| \leq |\kappa \times \kappa| \leq \kappa$. But this contradicts with the assumption that we made before. So there needs to exist a $y \in M_\xi$ such that $|f_4^{-1}(y)| \geq \kappa^+$. On the other hand, This means that at least $\kappa^+$ elements in the $A_\xi$ all have edges to a subset of $B_\xi$ with $\tau$ elements. Also $A_\xi \cap B_\xi = \emptyset$.This means there exist a complete bipartite $[\kappa^+, \tau]$ in $\mathscr{G}$. This fact contradicts with condition in the Lemma. Because of this reasons, we can conclude that $|A_\xi| \leq \kappa$. Thus, induction hypothesis is true for every $\xi \leq \omega$.
For the last part of the proof, we are going to show that $B_\omega$ is both $\tau - closed$ and $|B_\omega| \leq \kappa$. Second one is easier to prove . $B_\omega = \bigcup_{\zeta<\omega} A_\zeta$ also $|A_\zeta| \leq \kappa$ for every $\zeta < \omega$. Again using **Claim 2** we can conclude that $|B_\omega| = |\bigcup_{\zeta<\omega} A_\zeta| \leq |\kappa \times \omega| = \kappa$. Now, we going to prove $B_\omega$ is $\tau - closed$. If $x \in g \setminus B_\omega$ then $v(x, B_\omega, \mathscr{G}) = \bigcup_{\zeta<\omega} v(x, B_\xi, \mathscr{G})$ and $\tau(x, B_\xi, \mathscr{G}) < \tau$ for every $\zeta < \omega$. Considering that $\tau$ is finite and $\omega$ is infinite and regular it follows that there is a $\xi_0 < \omega$ such that $\tau(x, B_{\xi_0}, \mathscr{G}) = \tau(x, B_\omega, \mathscr{G}) < \tau$. It follows that $Clos(g', \mathscr{G}, \tau) \subseteq B_\omega$ and thus lemma is proven. ■

**Lemma 5**: Assume $|\omega| \leq \beta$ and $\tau < \beta$. If $Col(g_\xi) \leq \beta$ for every $\xi < \alpha$, then $Col(\mathscr{G}) \leq \beta$.
For every $\xi < \alpha$ let $<_\xi$ be the well ordering that satisfies the property of $Col(\mathscr{G}) = \delta \leq \beta$. If $x, y \in g$ let $x < y$ if only if $x \in g_\zeta, y \in g_\xi$ and either $\zeta < \xi$ or ($\zeta = \xi$ and $x <_\xi y$). By Lemma 1 $<$ is a well ordering of $g$. For every $y \in g_\xi$, $v(y, g| < y, \mathscr{G}) = v(y, h_\xi, \mathscr{G}) \cup v(y, g_\xi| <_\xi y, \mathscr{G})$. It follows from Lemma 2 that $|v(y, g| < y, \mathscr{G})| < |\tau^+ \cup \beta = \beta|$. This is because $\beta$ is an infinite cardinal and $\tau^+ \leq \beta$, so $<$ is a $\beta$-coloring of $g$. ■

**Definition 5:** Let $\mathscr{G}$ be a graph and $g$ denote its set of vertices and $G$ denotes its set of edges. Let us call $|g| = \alpha$. The relation $Col(\alpha, \beta, \gamma, \delta)$ is said to hold for if every graph $\mathscr{G}$, $|g| = \alpha$ and $Col(\mathscr{G}) > \beta$ implies that $\mathscr{G}$ contain a complete bipartite graph $[\gamma, \delta]$. Also, the relation $Chr(\alpha, \beta, \gamma, \delta)$ is said to hold if for every graph $\mathscr{G}$, $|g| = \alpha$ and $Chr(\mathscr{G}) > \beta$ implies that $\mathscr{G}$ contains a complete bipartite graph $[\gamma, \delta]$.

### 3.Proof of the Theorem

**Theorem:** Assume $Chr(\mathscr{G}) > \beta \geq |\omega|$. Then the relation $Chr(\alpha, \beta, \beta^+, \delta)$ is satisfied if $\delta < \omega$.

**Claim 3**: $Col(\alpha, \beta, \gamma, \delta)$ implies $Chr(\alpha, \beta, \gamma, \delta)$.
This is a direct result of Lemma 1. That's because if $Chr(\mathscr{G}) > \beta$ then $Col(\mathscr{G}) \geq Chr(\mathscr{G}) > \beta$.

**Main Proof:** We prove the theorem by transfinite induction on $\alpha$. If $|g| \leq \beta$ then $Col(\mathscr{G}) \leq \beta$. That is why $Col(\alpha', \beta, \gamma, \delta)$ holds for every $\alpha' \leq \alpha$ and for every $\gamma, \delta$. Assume that $\alpha > \beta$ and $Col(\alpha', \beta, \beta^+, \delta)$ holds for every $\alpha' < \alpha$ and assume that $\mathscr{G}$ is a graph with $|g| = \alpha$. We want to prove that if $Col(\mathscr{G}) > \beta$ then this graph contains a bipartite graph $[\beta^+, \delta]$. For this graph we are going to consider the sets $g_\xi$ that were defined in **Definition 3**. We choose $\tau$ to be $\delta^+$, since the value of $\tau$ was left open to be any cardinal.
We want to prove that $|g_\xi| < \alpha$ for every $\xi < \alpha$. To prove this, we will prove a stronger statement by transfinite induction.

**Lemma 6:** $|h_\xi| \leq$ max$(\beta, |\xi|)$ for every $\xi < \alpha$ (where max$(\kappa, \lambda)$ denotes the bigger cardinal number between $\kappa, \lambda$).

**Proof:** As previously said, we are going to use transfinite induction. Assume that the lemma is true for every $\zeta < \xi$ for some $\xi < \alpha$. We want to show that the lemma is true for $h_\xi$ as well. We are going to consider two cases.

**i)** $\xi$ is a limit ordinal.
$h_\xi = \bigcup_{\zeta<\xi} h_\zeta$ . It can be seen that $a \in h_\zeta$,for any $\zeta < \xi$ implies that $a \in \bigcup_{\zeta<\xi} g_\zeta$ as well. Also since $\xi$ is an infinite cardinal, if $\zeta < \xi$ then $\zeta + 1 < \xi$ satisfies as well. This suggest that $\bigcup_{\zeta<\xi} h_\zeta = \bigcup_{\zeta<\xi} h_{\zeta+1} \supseteq \bigcup_{\zeta<\xi} g_\zeta$. Combining this and above result we can conclude that $\bigcup_{\zeta<\xi} g_\zeta = \bigcup_{\zeta<\xi} h_\zeta$. For this case of the proof we are going to consider two different cases.

**i1)**$|\xi| \geq \beta$
We are going to show that $|\bigcup_{\zeta<\xi} g_\zeta| \leq |\xi|$ , from the induction hypothesis we know that $|h_\zeta| \leq |\xi|$ for every $\zeta < \xi$ . It is easy to see that $|g_\zeta| \leq |\xi|$ is also true for all $\zeta < \xi$. Using **Claim 2** , we know that $|\bigcup_{\zeta<\xi} g_\zeta| \leq |\xi \times \xi|$. Since $\xi \geq \beta \geq |\omega|$,$|\xi|$ is an infinite cardinal. So, $|\xi \times \xi| = ||\xi| \times |\xi|| = |\xi|$

**i2)** $\beta > |\xi|$
We are going to show that $|\bigcup_{\zeta<\xi} g_\zeta| \leq |\beta|$ , from the induction hypothesis we know that $|h_\zeta| \leq |\beta|$ for every $\zeta < \xi$ . It is easy to see that $|g_\zeta| \leq |\beta|$ is also true

for all $\zeta < \xi$. Using **Claim 2** , we know that $|\bigcup_{\zeta<\xi} g_\zeta| \leq |\xi \times \beta|$. Since, $\beta > \xi$ , $|\beta \times \beta| \leq |\xi \times \beta|$.Again using the fact that if $\beta \geq \omega$ , then $|\beta \times \beta| = \beta$ ; we conclude that $|\bigcup_{\zeta<\xi} g_\zeta| \leq |\beta|$. ■

**ii)** $\xi$ is a successor ordinal.
For this case, suppose $\xi = \zeta + 1$. Then by Lemma 3, Clos($h_\zeta \cup \{x_\zeta\}, \mathscr{G}, \tau) = h_{\zeta+1}$. From the induction hypothesis, $h_\zeta \leq$ max$(\beta, |\zeta|)$. If $|h_\zeta|$ is finite, then $|h_\zeta \cup \{x_\zeta\}|$ is finite as well. If $|h_\zeta|$ is infinite, then $|h_\zeta| = |h_\zeta \cup \{x_\zeta\}|$. In both the finite and infinite cases, $|h_\zeta \cup \{x_\zeta\}| \leq$ max$(\beta, |\zeta|) \leq$ max$(\beta, |\xi|)$. Let max$(\beta, |\xi|) = \gamma$. Then, $\gamma \geq |\omega|$. It follows from ***Lemma 4*** that $|h_\xi| \leq \gamma$.

Therefore, $|h_\xi| \leq$ max$(\beta, |\xi|)$ for every $\xi < \alpha$. ■

It is trivial to see that Lemma 6 implies that $|g_\xi| < \alpha$ for every $\xi < \alpha$. Suppose that our graph does not contain any complete bipartite graph $[\beta^+, \gamma]$. By considering the contrapositive of the desired statement at the beginning of this main proof, we can finish by showing that $Col(\mathscr{G}) \leq \beta$. If the entire graph does not contain any complete bipartite graph $[\beta^+, \gamma]$, no subgraph $\mathscr{G}(g_\xi)$ contains one as well. It follows from the main induction hypothesis that $Col(\mathscr{G}(g_\xi)) \leq \beta$ for every $\xi < \alpha$. Considering the fact that $\beta \geq |\omega| > \tau$, Lemma 5 implies that $Col(\mathscr{G}) \leq \beta$ . Hence, we get a contradiction with our assumption that $Col(\mathscr{G}) \geq Chr(\mathscr{G}) > \beta$ . Thus, theorem is true.

**4.Application of the Theorem**

In the proof above we suggested that if the chromatic number of a graph is not countable.($> |\omega|$) .Then there exist a complete bipartite graph of $[a, |\omega|^+]$ for every finite cardinal $a$. But how does this relate to our function problem ?
To understand this relation we need to interpret our problem in to graph theory language . We denote our vertex set $g$ of our graph to be the points in the 2 dimensional real space. Meaning that $g = \{(x, y) : x, y \in \mathbb{R}\}$ and our edge relation $R_g$ will be 'two different points in $g$ have an edge between them if only if they have a rational distance between them . Meaning that $\{a, b\} \in R_g \Leftrightarrow a \neq b \wedge d(a, b) \in \mathbb{Q}$. (as usual $d(a, b)$ denotes the Euclidean distance between the points $a, b$ in 2 dimensional euclidean space ). Let's call this graph $\mathcal{G}$. .Now it starts make sense to look whether our graph's chromatic number is countable or not. Now, our first step in the second part of the proof will prove that $Chr(\mathcal{G}) \leq |\omega|$ . Suppose contrary, then $Chr(\mathcal{G}) > |\omega|$ and from the theorem we proved we know that our graph $\mathcal{G}$ contains a complete bipartite graph $[a, |\omega|^+]$ for every finite positive integer $a$. We are going to prove that set of any two distinct vertices' common neighbours is countable in graph $\mathcal{G}$.

Suppose we have have two distinct points in real space, call them $a$ and $b$. It is a trivial fact that any two circles with different centers may intersect at most 2 points. So, if there is a common neighbour $X$ of vertices $a$ and $b$ in our graph , then this point needs to have a rational distance between both $a$ and $b$. Also, call this common neighbour set $C_{a,b}$. Now, we are going to define a injective function $f_5$ from $C_{a,b}$ to $Q^3$ . For any $X \in C_{a,b}$ let us denote $d(X, a) = q_{x,a}$ and $d(X, b) = q_{x,b}$ . There is two different situations
**i)** The circle with center $a$ and radius $q_{x,a}$ and the circle with center $b$ and radius $q_{x,b}$ intersect at exactly one point.
In this situation let $f_5(X) = (q_{x,a}, q_{x,b}, -1)$. (It is trivial that $(q_{x,a}, q_{x,b}, -1) \in \mathbb{Q}^3$

**ii)** The circle with center $a$ and radius $q_{x,a}$ and the circle with center $b$ and radius $q_{x,b}$ intersect at exactly two points.
In this case, these two points are reflections of each other with respect to line A, meaning they are on opposite sides of this line. Call one side as $S_1$ and other as $S_2$. If $X \in S_1$ then let $f_5(X) = (q_{x,a}, q_{x,b}, -1)$ and if $X \in S_2$ then let $f_5(X) = (q_{x,a}, q_{x,b}, -2)$.
It can be seen that every element in $C_{a,b}$ has a unique correspondence in the set $\mathbb{Q}^3$ under $f_5$. This means that $|C_{a,b}| \leq |\mathbb{Q}^3|$. Also $\mathbb{Q}^3$ is a countable set. Thus, we conclude that $|C_{a,b}| \leq |\omega|$. This suggest that any two vertices in our graph may have at most $|\omega|$ common neighbour which contradicts the existence of the complete graph $[|\omega|^+, a]$ for $a \leq 2$. Hence, $Chr(\mathcal{G})| \leq \omega|$.

Finally, we arrive to the last part of proof. We are going to use $Chr(\mathcal{G}) \leq |\omega|$ to finish the proof. By definition $Chr(\mathcal{G})$ is the least cardinal such that $g$ is the union $a$ sets ,where no two elements of same set are connected by an edge in $\mathcal{G}$. Suppose that for a such partition of $g$ . Let $M_g$ be the set of all sets in such partition. $\bigcup\limits_{S \in M_g} S = g$ . Considering the fact that, $Chr(\mathcal{G}) \leq |\omega|$ , there is a bijective $f_6$ function from some subset of $\mathbb{Z}^+$ ( trivially $|\mathbb{Z}^+| = |\omega|$) to the elements of $M_g$. Call this subset $K$ and also call $S_k = f_6(k)$ for every $k \in K$. So , we get $\bigcup\limits_{k \in K} S_k = g$. (where each $S_k$ is disjointed from others and doesn't include an edge in it.)
We choose our function $f$ from $\mathbb{R}^2$ to $\mathbb{Z}$ as follows.
For every $a \in \mathcal{G}$ , $a \in S_k$ for a unique $k \in K$ as well. We define our function f to $f(a) = k$ . So we can see that if any two distinct elements have a same image under $f$, this means they belong to same $S_k$ meaning that they don't have an edge in common. By the define of the graph $\mathcal{G}$ , this is equivalent to say that they have an irrational distance between them. So existence of a such function is true. ■

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